Real Estate Is Not Normal: A Fresh Look at Real Estate Return Distributions

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Abstract: Investment risk models with infinite variance provide a better description of distributions of individual property returns in the Russell-NCREIF data base over the period 1980 to 1992 than normally distributed risk models. Real estate investment risk is heteroscedastic, but the characteristic exponent of the investment risk function is constant across time and property type. Asset diversification is far less effective at reducing the impact of non-systematic investment risk on real estate portfolios than in the case of assets with normally distributed investment risk. Multi-risk factor portfolio allocation models based on measures of investment codependence from finite-variance statistics are ineffectual in the real estate context.

Key words: Asset-specific risk, return distributions, nonnormality, diversification, institutional investing

Prior to the 1980s, the commercial real estate equity sector was the domain of local entrepreneurs who viewed real estate investment as an exercise in business management. The real estate market at that time was a collection of loosely related regional submarkets, in which participants relied upon personal marketing, sales, and operating expertise to create real estate value over long investment horizons. Consistent with corporate capital budgeting practices of the time, investors analyzed prospective real estate purchases as stand-alone business opportunities, with little consideration for how new investments would interact with other properties or non-real estate assets in investor portfolios.

The emergence of corporate and public pension plans as major real estate equity investors in the 1980s brought a new investment perspective to the real estate sector. This new investor class viewed real estate as a vehicle for diversification to reduce risk in portfolios that consisted primarily of stocks and bonds. In recent years, pension plan consultants and institutional real estate investment managers have extended the concept to mean efficient diversification within the real estate sector as well as across major asset classes.

As presently viewed by most institutional real estate professionals, efficient diversification in real estate is achieved by adapting to commercial real estate the mean-variance portfolio allocation/optimization methodologies known collectively as Modern Portfolio Theory (MPT). These methodologies assume that investment risk can be modeled by decomposing risk into independent market-sector, market/systematic, and asset-specific components, each described by a separate normal distribution.

As MPT was taking shape in the 1960s, framers of the theory recognized that key assumptions underlying the models would have to be empirically tested: in particular, the assumption that asset investment risk can be modeled by normal distributions. The basic conceptual conclusion of MPT that asset-specific risk is a minor concern in portfolio theory compared to market/systematic and market-sector risk depends critically on this assumption. Expressed in the terminology of portfolio management, this means that the validity of the strategic assertion that the key issue in portfolio construction is asset allocation rather than asset selection depends on the nature of the distributions of the random variables assumed to model investment risk.

Empirical tests of the risk normality assumption using stock and bond market data were first conducted in the 1960s and continue to this day. As we discuss in the fourth section, these tests have raised strong doubts about the validity of the risk normality assumption. However, they also have raised doubts about the suitability of alternative distributions proposed as candidates for investment risk models.

Lacking evidence that mean-variance MPT methodologies are less accurate than alternatives, stock and bond investment managers have embraced MPT methodologies as compelling conceptual and tactical portfolio management tools.

In the same vein, institutional real estate managers and consultants have begun the extension of MPT to real estate investments by analogy with MPT stock and bond investment applications. Most real estate researchers followed the example of their colleagues in financial economics and simply assumed that real estate returns are normally distributed; virtually all researchers assumed that real estate return distributions have finite variance.[1] As with MPT stock and bond applications, managers who base portfolio decisions on these models often assert that the key issue in portfolio construction is real estate asset allocation rather than asset selection.

Now that individual institutional-grade property performance data are available from NCREIF, it is possible to test empirically the presumptions that property return distributions have finite variance and are Gaussian normal. The purpose of this study is to test whether property return distributions have finite variance, and to examine the implications of the test results for real estate portfolio construction, investment, and management.

The Statistical Shape of Investment Risk

A restricted test of the nature of property return distributions can be made by rational analysis without the need for observational data.[2] Assume for the moment that markets operate continuously, that asset investment risk functions for non-overlapping time intervals are independent, and that each asset risk distribution is unchanging over time. Then, for each positive integer *n*, the random variable describing the investment risk function for an asset over any specified time interval can be represented as the sum of *n* independent identically distributed (IID) random variables. For example, divide the interval into *n* nonoverlapping subintervals of

1 There are some exceptions, however. Myer and Webb (1990) tested this proposition for the quarterly returns on the Russell-NCREIF Property Index, and concluded that the Index residuals were samples from an infinite-variance stable Paretian distribution. Myer and Webb (1993) report that all of the Russell-NCREIF indices, except the Total Index and the Retail Index, are significantly fat tailed. Also, Liu *et al.* (1992) cast doubt on the normality assumption by noting the negative skewness of real estate portfolio systematic risk.

2 This discussion extends the one presented in Fama and Miller (1972), Chapter 6, Section IV.D.4, although this reasoning was familiar to economists and mathematicians in the late 19th Century. See Mandelbrot (1960) for a discussion of related economic analysis in the early 20th Century.

equal duration and express the original investment risk function as the sum of the *n* investment risk functions over the component subintervals.

Distributions of random variables that can be represented as the sum of an arbitrarily large number of independent identically distributed summands are known as infinitely divisible. The discussion in the preceding paragraph shows that the distribution of the investment risk function for an asset in a continuously operating market is infinitely divisible under the additional assumptions that investment risk is unchanging over time—i.e., investment risk depends on the length of the holding period, but not on when the holding period begins—and that investment risks for the same asset in non-overlapping time periods are independent.

The Central Limit Theorem of elementary probability and statistics strongly suggests that the only infinitely divisible probability distribution with finite variance is the normal distribution. Researchers have proven this result formally in probability theory using nonelementary methods.[3]

This leaves three primary sources of difficulty with the assertion that investment returns are normally distributed: (1) investment risk functions might not be independent for non-overlapping time intervals of the same duration, (2) investment risk functions might be different for non-overlapping time intervals of the same duration, and (3) investment risk functions might not have finite variance.[4]

If markets are efficient, then standard economic arguments show that investment risk functions over non-overlapping time intervals are independent. Thus, restricting this test to efficient markets, there can be only two explanations for the failure of investment risk functions to be normally distributed: investment risk changes with time (e.g., investment risk is heteroscedastic), or investment risk functions have non-normal infinitely divisible distributions.

For economic risk models to have practical value, researchers must be able to infer current investment risk from historical investment returns. Although large quantities of data and modern data-processing technology can support the development of time-varying risk models, prior to the general availability of computers the risk models of choice for researchers were unvarying over moderately large time intervals. Thus, it should not be surprising that the framers of MPT were extremely interested in investment risk models that were: (1) independent for non-overlapping time intervals, (2) unchanging over time, and (3) infinitely divisible with infinite variance rather than normally distributed.

To framers of MPT, infinite-variance infinitely divisible distributions had conceptual economic appeal as well as theoretical statistical appeal because real-world investment return series appeared to resemble sample returns from these distributions more closely than they resembled sample returns from normal distributions. Sharpe (1970) and Fama and Miller (1972) observed that normal distributions were crucial to implementation of mean-variance techniques and that much of the then-available evidence indicated that stable non-normal distributions provided a better fit to sample distributions of asset and portfolio returns than normally distributed models.

These authors were, for the most part, relying on observations made by economists in the early part of the 20th Century. Mandelbrot (1963a, pp. 394-5), observed that "he empirical

3 See Gnedenko and Kolmogorov (1954), Chapter 7.

4 A fourth caveat is that this argument should apply to continuously compounded returns, i.e., the logarithms of asset value change over finite time intervals (see Mandelbrot (1963b) and Fama (1965a)), but this substitution does not affect the remaining argument.

distributions of price changes are usually too 'peaked" to be…samples from Gaussian populations," and added that this characteristic had been noted in publications as early as 1915. He also pointed out that "there are typically so many outliers…. The tails…are in fact so extraordinarily long that the sample second moments [variances] typically vary in an erratic fashion. For example, the second moment…in [a sample distribution of monthly wool prices] does not tend to any limit even though the sample size is enormous by economic standards, and even though the series to which it applies is presumably stationary." The obvious inference is that the sample variances do not converge because the true variance does not exist—at least, as a finite number—for this distribution.

As Sharpe (1970) observed, the primary conceptual difference between infinite-variance distributions and finite-variance distributions is that extreme sample values occur with greater frequency in the infinite-variance case. This means that sudden dislocations in real-world markets can be modeled by a single infinitely divisible random variable with infinite variance.

Stable Distributions

The French mathematician Paul Levy introduced and characterized a class of probability distributions known as stable, which Levy and the Russian mathematician A.Y. Khintchine proved coincides with the class of infinitely divisible distributions. See Gnedenko and Kolmogorov (1954, Chapter 7).

As discussed in the previous section, normal distributions are stable and are the only stable distributions with finite variance. Other examples of stable distributions are the well-known Cauchy distributions. The normal and Cauchy distributions are the only stable distributions for which probability densities can be expressed in closed form in terms of elementary mathematical functions.

Although most stable distributions and their probability densities cannot be described in closed mathematical form, their characteristic functions—and the logarithms of the characteristic functions—can be written in closed form. The log characteristic functions of stable distributions have the following form:

$$\psi(t) = \begin{cases} i\delta t - |ct|^{\alpha}\left[1 - i\beta \operatorname{sgn}(t)\tan(\pi\alpha/2)\right], & \text{for } \alpha \neq 1 \\ i\delta t - |ct|\left[1 + i\beta(2/\pi)\operatorname{sgn}(t)\log|t|\right], & \text{for } \alpha = 1 \end{cases} \quad (1)$$

The four parameters α, β, c, and δ in Equation (1) completely characterize the distribution.

The characteristic exponent α lies in the half-open interval (0,2] and measures the rate at which the tails of the density function decline to zero. The larger the value of the characteristic exponent α, the faster the tails shrink toward zero. When α=2.0, the distribution is normal.

A stable distribution with characteristic exponent α has moments of order $<\alpha$, and does not have moments of order $>\alpha$. While the means (first moments) of stable distributions with characteristic exponents α>1.0 do exist, variances (second moments) do not exist—i.e., are infinite—for those distributions with characteristic exponents α<2.0.

The skewness parameter β lies in the closed interval [-1,1], and is a measure of the asymmetry of the distribution. The significance of β can be described as follows: let x_0 be the point of maximum likelihood for the probability distribution, i.e., the point where the probability density assumes its highest value (equivalently, this is the unique point in the domain of the distribution at which the slope of the probability density is zero). If β is positive, then there is more area under the density function to the right of x_0 than there is to the left of x_0 which implies that the probability is greater than one-half that a sample value from the distribution is larger than x_0; equivalently, the density function is skewed to the right.[5] Similarly, if β is negative, then the majority of the area under the density function is to the left of x_0, which implies that the probability is greater than one-half that a sample value from the distribution is smaller than x_0. If and only if β=0, then x_0 coincides with the distribution median, and the distribution is referred to as symmetric. It is easy to verify that a stable distribution is symmetric in this sense if, and only if, the graph of its density function is symmetric in the usual reflective sense about the vertical line through x_0.

The scale parameter c lies in the open interval (0, ∞), and is a measure of the spread of the distribution. If α=2.0, the scale parameter c is directly proportional to the standard deviation: $c = \sigma / \sqrt{2}$. However, the scale parameter c is finite for all stable distributions, despite the fact that the standard deviation is infinite for all α<2.0. Thus, the scale parameter c can be regarded as a generalization of the standard deviation.

The location parameter δ may be any real number, and is a rough measure of the midpoint of the distribution. A change in δ simply shifts the graph of the distribution left or right, hence the term "location." If the distribution is symmetric, then δ coincides with the point of maximum likelihood for the distribution, which in turn coincides with the median. If, in addition, α>1.0, so that the mean of the distribution exists, then these three values also coincide with the mean.

The reason that distributions of the form of Equation (1) are referred to as stable can be described intuitively in terms of the parameters a α, β,, c, and δ. Assume that f and g are two independent random variables with the same domain of definition, that f and g both have stable distributions, and that the two stable distributions have the same characteristic exponent α_0 and the same skewness β_0. Then the definition of stability is designed to imply that the distribution for the random variable $f + g$ is also stable, and that the characteristic exponent and skewness of the distribution for $f + g$ are the same as the characteristic exponent and skewness of the distributions for f and g (namely, α_0 and β_0 respectively). Note that this assertion does not assume any relation between the scale or location parameters for f and g It follows nonetheless that a simple relation exists between the scale parameter of $f + g$ and the scale parameters of f and g, and similarly between the location parameter of $f + g$ and the location parameters of f and g:

$$c_{f+g}^{\alpha_0} = c_f^{\alpha_0} + c_g^{\alpha_0} \tag{2}$$

$$\delta_{f+g} = \delta_f + \delta_g \tag{3}$$

Equation (2) generalizes the relation between the standard deviation of the sum of two random variables and the individual standard deviations of the two random variables. Equation (3) extends the general principle that the expectation of the sum of two random variables is the sum of the individual expectations.

Similarly, if k is any non-zero real number and f is a stable infinite-variance random variable with stable parameters as described above, then k•f is also a stable infinite-variance random variable and the stable parameters of k•f are : α_0, β_0, $|kc_f|$, and k•δ.

[5] The research literature on stable distributions contains several inconsistencies in the definition of the skewness parameter β. This difficulty had its origin in the fact that, because of the way early versions of Equation (1) were specified, asymmetric stable distributions had skewness that appeared intuitively negative for positive values of β, and vice versa. McCulloch [26] discusses this problem in detail.

The closer the characteristic exponent α is to the upper limit of the permissible range—i.e., the value 2.0—the less significance the skewness has in terms of shifting the shape of the distribution away from the corresponding symmetric distribution with the same parameters α, c, and δ. At the limit α=2.0, the skewness parameter β becomes irrelevant and all stable distributions are symmetric. This is seen by referring to Equation (1) and noting that $i\beta \operatorname{sgn}(t)\tan(\pi\alpha/2)=0$ for all values of β if α=2.0, since $\tan(\pi)=0$. The Cauchy distributions with α=1.0 mentioned above are also symmetric and stable.

In elementary portfolio theory where investment risk is modeled by normal distributions, investment return and risk (as measured by the standard deviation of return) for individual assets are represented in terms of points in two-dimensional Cartesian geometry. Once this identification is made, economic hypotheses can be combined with two-dimensional geometry to derive results such as the Sharpe-Lintner Capital Asset Pricing Model (CAPM) and computational algorithms for determination of the efficient portfolio set.

In order to represent investment return and risk in terms of Cartesian geometry, it is not necessary to make the assumption that all asset investment risk over the specified time interval is normally distributed, only that all asset risk distributions are stable with the same characteristic exponent α and skewness parameter β. Despite the fact that the standard deviation is infinite if the characteristic exponent α is less than 2.0, the conceptual geometric representation of return and risk is unaltered if the measurement of investment risk is redefined as the scale parameter c of the asset risk distribution.

Despite the availability of a two-dimensional representation of asset return and risk in the stable infinite-variance case, analogs to the CAPM or alternative asset-pricing models have not yet been derived in the infinite-variance setting.[6] The difficulty in rederiving MPT in the case of general stable risk functions lies in the fact that no reasonable analogs of codependence measurements such as correlation have been developed for stable infinite-variance distributions. In any case, this is not the key issue in the investigation of stable infinite-variance risk functions from the perspective of portfolio management; the key issue is how the relative importance of portfolio diversification versus asset selection changes as the characteristic exponent of asset risk gets further away from 2.0.

Related Research

Benoit Mandelbrot was the first investigator during the MPT era to raise the possibility that stable infinite-variance distributions are the appropriate models for investment risk. In a series of articles published in the early to mid-1960s, Mandelbrot suggested stable infinite-variance distributions were more appropriate models than normal distributions for various income and commodity price series. Mandelbrot (1960) suggested that stable infinite-variance distributions could be applied to a wide variety of stochastic economic problems. He also pointed out that this suggestion had already been made in a weaker form by V. Pareto at the end of the 19th Century,

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6 Fama (1965b) investigated portfolio theory in the case of general stable risk functions under the additional assumption that the standard market model is applicable. However, he conceded that a derivation of the market model in this setting was beyond the state of statistical technology at that time. The observation is still valid. Samuelson (1967) derived a system of equations with constraints to determine the efficient portfolio set in the case of stable infinite-variance risk. However, Blume (1970) observed that Samuelson did not indicate how this system of equations should be solved.

and suggested the term “Paretian” to describe general distributions with characteristic exponent α between 0 and 2.[7]

Mandelbrot (1963a) presented an in-depth survey of stable distributions and a blueprint for applications he envisioned. Specifically, he discussed recasting the turn-of-the-century work of Bachelier on random walk models for commodity and security prices in terms of stable infinite-variance distributions, presented a mathematical summary of the theory of stable distributions, and presented an application to logarithms of daily and monthly cotton price changes during various subintervals of the period 1816 to 1958. This application was limited in precision, because at the time only simple graphical techniques were available to estimate the parameters of a stable distribution from a sample set.

Mandelbrot concluded that a stationary, stable distribution could not provide a good fit to the observed data on cotton prices. However, by assuming that the scale parameter changed over time (i.e., was heteroscedastic), he was able to fit a symmetric stable infinite-variance distribution to the data with a time-invariant characteristic exponent α of approximately 1.7 (1963a, Section III.C). Interestingly, Mandelbrot observed that the data appeared slightly positively skewed.[8] Perhaps due to the limitations of his analytical techniques—which apparently did not include a way to estimate skewness—Mandelbrot did not emphasize that data skewness provided additional support for the hypothesis that stable infinite-variance distributions are the appropriate models for investment risk. Instead, he insisted that the degree of skewness was small, and asserted accordingly that the sample distribution could be modeled by a symmetric stable distribution.

Mandelbrot (1963a, Section VI.B) also discussed the difference between continuous time stochastic processes in the normal and the stable infinite-variance cases. He pointed out that continuous time normal (i.e., diffusion) stochastic processes have continuous sample paths, but that continuous time stable infinite-variance stochastic processes have sample paths that are discontinuous almost everywhere. He speculated that this did not appeal to early 20th Century economists, and was one of the primary reasons that the economics establishment rejected stable infinite-variance candidates for economic risk models during the first round of application proposals in the 1920s and 1930s. [9]

Mandelbrot (1963b) and Fama (1963) discussed micromarket mechanisms that can produce stable infinite-variance distributions for investment risk over finite time intervals.[10] They showed

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7 The work of Pareto preceded the work of Levy and Khintchine; Pareto did not have access to the theory of stable distributions or Generalized Central Limit Theorems.

8 In the article, Mandelbrot actually remarked that skewness "…takes a small negative value." However, as discussed in Note 5, the definition of the skewness parameter used in Mandelbrot ((1963a) was the negative of the definition now used by most investigators.

9 McCulloch (1978) examined computer-generated stochastic processes in which the diffusion term in the stochastic equation is replaced by the infinitesimal generator of a symmetric stable infinite-variance process. He concluded that computer-generated series with characteristic exponents α intermediate between the characteristic exponents for the normal and Cauchy distributions (i.e., α in the neighborhood of 1.5—midway between α=2.0 and α=1.0) resemble observed price series in the financial markets more closely than simulations generated by either the normal or Cauchy distribution.

10 The argument originated in Mandelbrot (1963b), although the discussion in Fama (1963) is more readable. This is conceptually the same argument presented for the normal risk model in Section 2 of the present article, applying the Generalized Central Limit Theorem for infinite-variance distributions in place of the ordinary Central Limit Theorem.

that, if new market information produces micromarket price changes described by changing—not necessarily stable—Paretian distributions with the same characteristic exponent α, then the Generalized Central Limit Theorem implies under very general conditions that investment risk is stable with the characteristic exponent α over finite investment horizons.

Fama (1965) conducted a comprehensive statistical analysis of pricing efficiency in the U.S. stock market, together with an examination of the shape of stock investment risk. Testing logarithms of daily price change series for 30 stocks (pp. 62-8, results summarized in Exhibit 9), he concluded that the stable infinite-variance risk model fit the data better than a normal or a mixture-of-normals model. Furthermore, he concluded that the data were consistent with the assumption that investment risk functions for all 30 stocks had the same characteristic exponent α, and that the value of α was approximately 1.9—more correctly, in the interval [1.85, 1.95]. Fama used three techniques to estimate α: the graphical technique used in Mandelbrot (1963a) to examine cotton price changes, and two others that Fama indicated were probably less accurate. The three techniques produced consistent results. With related techniques, Blume (1970) concluded that the best estimate for a common value for the characteristic exponent for the stock market should be in the interval [1.7, 1.8].

In a search for improved parameter estimation techniques, Fama and Roll (1968, 1971) examined the effect of stable distribution parameter values on sample distribution order statistics.[11] They introduced a set of simple asymptotically normal parameter estimators for the symmetric case (i.e.,$\beta = 0$) for the restricted range of characteristic exponents $1.0 \leq \alpha \leq 2.0$, along with standard error estimates derived by Monte Carlo simulation. This was a significant technological advance, because it provided researchers with the capability to make straightforward tests of the stable infinite-variance investment risk model on virtually any set of investment returns or price changes. However, the constraint to symmetric stable models was also a major limitation, because it was becoming widely recognized that skewness is present in most economic return distributions.

Even before the second Fama-Roll article appeared, Roll (1970, Chapter 6), applied their new estimators to test the stable infinite-variance risk model on weekly changes in the yield curve for T-bill maturities under six months. He found internally consistent sample characteristic exponents α in the interval [1.12, 1.53] for maturities up to three months, characteristic exponents α ranging as low as 1.0 for longer maturities, and scale parameters c that declined as maturities lengthened. Since the Fama-Roll estimators assume symmetric distributions, Roll applied a standard skewness test to the samples (pp. 74-5) and concluded that the assumption of symmetry was consistent with the data. He also discussed the application of goodness-of-fit tests (pp. 75-81) to compare actual data groupings with predictions of the fitted symmetric stable distributions. Although most of the χ^2 values for these groupings exceeded the critical value at the 0.01 significance level, Roll pointed out that market-constrained discreteness of permissible yield values destroyed the ability of the test to discriminate between sample distributions. He concluded that symmetric stable infinite-variance distributions fit the data better than any other continuous distributions.

In an examination of government security liquidity premia, McCulloch (1975) extended the test of interest rate changes to maturities as long as 30 years, and found sample values of α similar

11 See Kendall and Stuart (1963-68), Chapters 14, 31, and 32.

to those observed by Roll. However, he was unable to reject an alternative hypothesis that the apparent non-normality of interest rate changes was due to heteroscedasticity.

Also before the appearance of the second Fama-Roll article, Teichmoeller (1971) applied the Fama-Roll estimators to test the characteristic exponents of logarithmic price series changes for 30 NYSE stocks—in essence, an update of the characteristic exponent test in Fama (1965a) with better statistical technology. The mean value of the 30 α estimates was 1.64, with a standard deviation of 0.18—so 2.0 is at the extreme upper end of the 2σ confidence band. Interestingly, while three of the sample α values equaled 2.0, Teichmoeller's examination of the data showed large numbers of days in which these three stocks did not change price. He observed that this behavior complicates the applicability of the Fama-Roll estimator for the characteristic exponent and inflates the values of α it generates.

Officer (1972) examined investment risk for daily and monthly stock returns. He concluded that empirical stock market return distributions have thicker tails than normal distributions, but that the sample distributions also display some of the characteristics of distributions with finite variance. Barnea and Downes (1973) replicated the Teichmoeller study using a different data set of daily stock price changes. While accepting Teichmoeller's methodology, they criticized the Teichmoeller data set for containing four preferred stock issues and common stocks of three investment companies. Barnea and Downes observed that the value of the characteristic exponent appeared to be a function of the stock issue, and that the distributions of some stocks did not seem to be stable. However, the results were inconclusive.

Leitch and Paulson (1975) introduced a complicated numerical procedure based on the calculus of variations to estimate stable Paretian distribution parameters for arbitrary $-1.0 \leq \beta \leq 1.0$. They applied the technique to monthly log price changes of 20 NYSE common stocks. They determined that most stock investment risk distributions were highly skewed, though the skewness parameter β could be either positive or negative for individual stocks. They also compared their parameter estimates of α and c with the corresponding parameters calculated from Fama-Roll estimators under the assumption that $\beta=0$, and concluded that the Fama-Roll estimators work very well for $\alpha \geq 1.6$.

The appeal of stable infinite-variance distributions was dealt a blow by the appearance of Blattberg and Gonedes (1974). Ignoring the theoretical economic argument that implies that investment risk should have a stable distribution, they proposed symmetric finite-variance alternatives for investment risk selected from the set of Student's t-distributions with low degrees of freedom— based upon their analysis, typically between 2 and 8 degrees of freedom. The data analysis in their article compared the fit of Student's t and symmetric stable infinite-variance distributions to daily stock market return series.[12] While Blattberg and Gonedes conceded that Student's t-distribution did not fully account for observed properties of the sample series, they concluded that Student's t-distributions provided a better empirical description than the symmetric stable distributions.[13] Interestingly, their estimated values of α ranged between 1.45

[12] Blattberg and Gonedes analyzed actual daily returns instead of the logarithms of daily returns. They pointed out that for the small return values encountered in daily data, $\ln(1+r_t)/r_t$ is nearly identical to 1, so that little is lost by using discrete returns instead of continuously compounded returns.

[13] When Blattberg and Gonedes tried fitting their model to sample returns, they found no stationarity in any of the parameter estimates that would suggest some predictive benefit from using the model. Furthermore, as we will show later, real estate return distributions are similar to return distributions of other asset classes in that they exhibit periods of negative or positive skewness (see, for example, Turner and

and 1.87 with a mean of 1.65, which were consistent with earlier estimates of characteristic exponents based on daily individual stock returns such as Teichmoeller (1971).

The appearance of Blattberg and Gonedes (1974) and several related studies questioning the appropriateness of infinite-variance investment risk functions, coupled with the extreme interest Wall Street and pension sponsors were showing in computerized implementations of MPT, signaled the end of interest by mainstream financial economists in stable infinite-variance investment risk models. By the late 1970s, Fama, Miller, Roll, Sharpe, and Samuelson had returned to the MPT fold, and Mandelbrot had gone on to investigations in chaos theory towards which stable Paretian distributions had pointed the way. However, a few statisticians and economists continued to investigate stable distributions, economic applications, and related questions.

Simkowitz and Beedles (1980) tested continuously compounded monthly return series of the Dow-Jones Industrial Stocks for skewness. They concluded that significant skewness of returns is the rule rather than the exception, that both positively and negatively skewed distributions occur, but that positive skewness occurs with significantly greater frequency than negative skewness.

Morgan (1976) and Tauchen and Pitts (1983) investigated heteroscedasticity of continuously compounded returns in the stock and futures markets, respectively. Morgan showed that the volatility of both 4-day and monthly returns for a sample of NYSE stocks is directly related to the volume of trading activity. Tauchen and Pitts demonstrated a similar relation between volatility and trading activity for daily returns on 90-day T-bill futures, and went on to develop a nonlinear model to explain changes in T-bill futures volatility.

Recently, in an article of major significance, McCulloch (1986) developed a set of simple asymptotically normal estimators for stable Paretian distribution parameters based upon order statistics that appear to be extensions of the Fama-Roll estimators. Because McCulloch developed estimators for all four defining parameters of a stable distribution, his estimators are applicable to both symmetric and asymmetric stable distributions. Additionally, the estimators apply to distributions with characteristic exponents in the interval $0.6 \leq \alpha \leq 2.0$, an expansion of the range of characteristic exponential values to which the Fama-Roll estimators apply. Finally, McCulloch provides simple techniques to estimate the standard errors of his parameter estimators, based upon the asymptotic normality of the estimators.

In addition, since 1975 a number of investigations in the statistics literature have shown how to obtain maximum likelihood estimators for stable distribution parameters. Like the Lietch and Paulson (1975) estimators, these maximum likelihood estimators are complicated and numerically intensive to implement. While they are theoretically more accurate than the McCulloch estimators, in most economic applications the marginal increase in accuracy over the McCulloch estimators is more than outweighed by the far greater inconvenience of implementation. The only applications in which these estimators are likely to be important are those involving relatively small data sets having sample characteristic exponents close to 2.0.

Unlike the situation in the financial economics and statistics literature, little has appeared in the real estate literature questioning the normality assumption for investment risk. A notable exception is Myer and Webb (1990 and 1993, cf. Note 1). In addition, Liu *et al.* (1992) have

Weigel (1992) for the case of stocks). Note that Student's t-distributions are symmetric, which introduces additional concern over the applicability of the Blattberg and Gonedes model.

shown recently that systematic risk for appraisal-based portfolio returns is negatively skewed, casting additional doubt on the normality assumption for real estate investment risk functions.

Data Description

Until recently, individual property returns from large, well-diversified pools of unleveraged institutional-grade U.S. commercial real estate were not accessible to most researchers. Only aggregate U.S. commercial real estate index data were available, and relatively little of that—most notably, the Russell-NCREIF Property Index and the various subindices compiled along major geographical region, property type, and combined region and property type dimensions.[14]

In 1992, due in part to the demands of pension fund sponsors and managers for more complete reporting and their desire to encourage in-depth real estate investment research of the kind possible only with disaggregated property data, the returns on the individual properties that comprise the various Russell-NCREIF indices became available.

Two non-overlapping sets of individual property returns are available: returns on unleveraged properties in the Russell-NCREIF Property Index, and deleveraged returns on leveraged properties submitted by members of NCREIF but not incorporated in the Russell-NCREIF Property Index.[15] Both the unleveraged and the deleveraged data sets contain net operating income, capital improvement, and carrying value figures that permit computation of total returns without the debt component. Thus, it is reasonable to assume that the sample returns obtained by deleveraging leveraged property returns are equivalent to unleveraged returns.

Some NCREIF members have expressed concern over the interpretation of deleveraged property returns. In general, these concerns revolve around the reported values of properties in declining markets, in which they suggest that reported values may have been propped up artificially to avoid reporting technical defaults under mortgage covenants relating to loan-to-value ratios. However, our comparison of unleveraged and deleveraged sample return distributions indicated no evidence of differences in the descriptive parameters for the various pairs of distributions, so we consolidated the unleveraged and deleveraged data sets to enlarge our sample sizes.[16]

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14 The most widely cited performance measure of institutional-grade real estate equity is the Russell-NCREIF Property Index. This industry-wide benchmark has existed since December 1977. As of December 1992, the Index combined the performance results for 1,892 unleveraged properties (including apartment and hotel properties) with an aggregate market value of approximately $24 billion.

15 The number of properties in the Russell-NCREIF combined data base for the years 1980 to 1992 identified as being office, retail, warehouse, and research & development are shown in Exhibit 6. In the Russell-NCREIF data bases, annual total time-weighted returns are computed by chain-linking quarterly time-weighted returns. We utilize these annual total time-weighted returns in this paper. The formula for quarterly time-weighted returns is:

$$\textit{Total Return} = (EMV - BMV + PS - CI + NI) / (BMV - 0.5\, PS + 0.5\, CI - 0.33\, NI)$$

where *EMV* is the ending market value for the quarter, *BMV* is the beginning market value for the quarter, *PS* is partial sales proceeds, *CI* is capital improvements made in the quarter, and *NI* is net property income in the quarter. Partial sales and capital improvements are assumed to occur in mid-quarter, while net income is assumed to be received monthly. These assumptions account for the coefficients on the variables in the denominator.

16 Titman and Torous (1989, pp. 345-6 and 349-50) asserted that deleveraging real estate equity returns is much more feasible and straightforward in general than would be the case for a leveraged asset with a more complicated capital structure such as the stock of a NYSE-listed company.

Reported returns are based on income and asset value changes (i.e., capital gains) as determined by appraisal. Both quarterly and annual returns are available, but we use only the annual total returns provided by The Frank Russell Company. Outside or third-party appraisals on most properties are conducted on an annual basis, which means that reported investment returns for three of the quarters each year are predicated on different criteria than returns for the remaining quarter. Some researchers have remarked that this distinction implies the superiority of annual returns over quarterly returns for investment analysis studies.[17] In fact, until recently many annual reappraisals took place during the fourth quarter of each calendar year, which has enabled some of these researchers to observe evidence of the distinction between quarterly reporting criteria in the time series behavior of aggregated quarterly returns on the various Russell-NCREIF indices.

Many researchers have long expressed a preference for continuously compounded returns over discrete risk in studies of the shape of investment risk.[18] There are four main justifications for this preference: (1) in many investment situations asset return is completely determined by asset price change, and logarithms of price changes have been the preferred data format for the study of price change distributions; (2) studies that depend on detailed examinations or comparisons of distributional tails are more easily conducted if the negative tails of the distributions are not truncated at -100%; (3) for return values less than 15% in magnitude, the difference between discrete and continuously compounded returns is negligible; and (4) for continuously compounded returns, the distribution of nominal returns is identical to the distribution of real returns except for the value of the location parameter δ.[19]

Combining the above considerations, the data set selected for this study has been chosen as large as possible by the inclusion of both unleveraged and deleveraged annual total returns on the properties in the Russell-NCREIF combined data bases for the years 1980-1992.[20] To permit comparisons of distributions by property type, the most commonly reported property types have been included: office, retail, warehouse, and research and development/office (R&D).[21]

Before beginning the data analysis, each discrete annual sample return r_t in the Russell-NCREIF combined data bases has been replaced with its continuously compounded equivalent, $\ln(1+r_t)$. Only properties having four quarters of data in a given calendar year have been included.

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17 See, for example, Giliberto (1990), p. 261, Graff and Cashdan (1990), p. 82, Gyourko and Keim (1992), p. 459, and Wheaton and Torto (1989), p. 442.

18 See Fama (1963), pp. 45-6.

19 Each locational parameter (e.g., median or mean) for a distribution of continuously compounded nominal returns is easily compared with the corresponding locational parameter for the distribution of corresponding real returns: the nominal return locational parameter is greater than the real return locational parameter by the continuously compounded change in the Consumer Price Index (CPI) for the period over which the return is measured. In other words, the conversion of the return distribution from nominal to real involves merely the subtraction of the logarithm of the corresponding periodic rate of inflation.

20 Prior to 1980, there were not enough properties in the Russell-NCREIF data bases to make meaningful estimates of the parameters that define stable distributions or, more generally, to discriminate between finite-variance and infinite-variance distributions.

21 Apartment and hotel properties were too few in number—especially in the early 1980s—to distinguish between normal and stable infinite-variance return distributions.

Real Estate Return Model

A statistical comparison of the data in the Russell-NCREIF Regional Property Type Subindices reveals significant differences among the annual returns for the various subindices. In our real estate market model we assume that these account for all the differences in expected individual property returns, i.e., that expected variations in annual property returns due to differences in property type account for all of the differences in expected unleveraged and deleveraged returns on properties in the Russell-NCREIF combined data base.[22]

More precisely, we assume that the observed annual total return on each commercial property *p* during the calendar year *t* is of the following form:

$$R_t(p) = \mu_t(h(p)) + \varepsilon_t(p) \qquad (4)$$

where $h()$ is the property type (office, retail, warehouse, or R&D), $\mu_t()$ is the expected total return during year *t* as a function of property type, and $\varepsilon_t(p)$ is a stable (possibly, infinite-variance) random variable. In addition, we assume that, for each $t \geq 1980$, the $\varepsilon_t()$ are independent identically distributed random variables with characteristic exponent $\alpha_t > 1.0$ and zero mean, and that $\varepsilon_{t_1}(p_i)$ and $\varepsilon_{t_2}(p_j)$ are independent for all $t_1 \neq t_2$ and all *i* and *j*.[23]

Under these assumptions, the random variable $\varepsilon_t(p)$ corresponds to the asset-specific investment risk of property *p* during period *t*, while the systematic and market sector real estate risk is described by the function $\mu_t(h(\ ))$.

This model implies that two properties of the same type have: (1) the same expected return, and (2) the same investment risk distribution. At first glance this seems quite different from stock market return/risk analysis. Studies of stock market return series suggest that the common stocks of two corporations engaged in the same general economic activities can display very different return/risk profiles. However, the two corporations are also likely to have distinct capital structures, which in turn implies that the two common stock issues represent different economic slices of the same kind of economic pie. Because corporate capital structures can be complex, the widely varying forms and provisions of corporate debt make it impractical to deleverage common stock returns on a routine basis. Nonetheless, it is reasonable to expect that deleveraged *ex ante* common stock returns and investment risk functions would be virtually identical for corporations engaged in the same types of business activities.

We acknowledge that the assumptions of this model differ considerably from the multi-dimensional normal MPT risk factor models currently popular with institutional real estate managers. However, we make the following two observations: (1) assumptions of normality and finite variance of real estate investment risk have rarely been subjected to rigorous empirical examination in the research literature, and the few studies that have been conducted (e.g., articles cited earlier: Myer and Webb (1990 and 1993), and Liu *et al.* (1992)) do not support the multi-dimensional normality assumption; and (2) if stable infinite-variance random variables are the

22 Alternatively, we could have broken down the returns by major geographic region. We believe that property type, however, is the superior cut, because it is more likely that investment characteristics of commercial property differ for properties with different drivers of economic performance than that investment characteristics differ for properties with the same economic functional applications situated in different locales. The free flow of institutional real estate investment capital across the country—as contrasted with an earlier era in which capital and investment decisions were more local in nature—will tend to homogenize transient differences in investment characteristics across geographical regions for property of the same type.

23 The assumption that $\alpha_t > 1.0$ guarantees that the mean of $\varepsilon_t(p)$ exists.

appropriate models for real estate investment risk, then the sample correlations that provide the rationale for multi-dimensional MPT return/risk models are not statistical estimators for any true measurements of codependence between real estate returns. These points call into question the empirical justifications customarily provided for multi-dimensional real estate risk factor models popular with institutional real estate managers.[24]

We do not assert that multi-dimensional factor models are inappropriate for real estate investment risk analysis. However, we do assert that most researchers and managers have not applied statistical tests properly to verify the applicability of their models to real estate portfolio management.[25]

Standard operating procedure in empirical real estate research has been to assume the normal probability distribution of asset-specific risk as an act of faith, and then apply statistical techniques to obtain descriptions of systematic and market-sector risk. By contrast, our tests will examine asset-specific investment risk under the assumptions of our model, with the objectives of (1) confirming or rejecting real-world applicability of the model, and (2) obtaining additional statistical information about the likely shape of real estate investment risk. In particular, the focus of this investigation is the test of a model for the distributional form of $\varepsilon_t(p)$; we will not propose a time-series model for $\mu_t(h(p))$ (i.e., a model for systematic or market-sector risk). However, we point out the practical difficulty of testing such a model: in contrast to a test of our model for asset-specific risk, for which we have available more than 13,000 sample values, any test of a model for systematic or market-sector risk can call on a data base of at most 52 sample values—four annual sector means per year over the thirteen-year sample period.

Since the concern of this article is with the distributional form of asset-specific risk, it is not necessary to address questions that have been raised recently about whether the Russell-NCREIF data base is a suitable proxy for the (unobservable) commercial real estate market; this issue has implications for tests of statistical models for systematic and market-sector risk components, but not for tests of models for asset-specific/diversifiable risk.

Tests and Results

Exhibits 1a to 5a show the distributions of continuously compounded annual total returns for the years 1980-92: (1) in the aggregate, and (2) by individual property type. Superimposed upon the sample histograms are normal densities with the corresponding means and standard deviations. In each case, the sample density function is more peaked near the mean than the corresponding normal density, has weaker shoulders and fatter tails (i.e., is leptokurtotic), and is negatively skewed. These distinctions can be seen more clearly in the graphs of the differences between each sample density and the corresponding normal density, Exhibits 1b to 5b.

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24 Even if real estate investment risk were normally distributed, we contend that institutional managers routinely make portfolio decisions on the basis of investment codependence data, while ignoring theoretical limitations on the accuracy of that data. This subject will be examined carefully in Graff and Young (1994).

25 In fact, recalling the discussion in the previous section, the preferability of annual return data over quarterly data implies that models for systematic risk or time-series behavior for individual property returns can be tested on data sets containing at most 13 sample values. By contrast, our cross-sectional annual return residuals average around 1,000 samples, suggesting that far more discriminatory statistical tests can be conducted on models for cross-sectional return residuals than on models for systematic risk or individual asset time-series behavior.

Before fitting stable distributions to the sample data, we corrected for possible extraneous data dispersion due to changing expected return by reducing each annual return by the corresponding sample mean for that calendar year and property type (cf. Equation (4)). The means are shown in Exhibit 6 for purposes of completeness, but will not be needed in the subsequent discussion.

We used the methodology of McCulloch (1986) to fit a stable distribution to each set of residuals decomposed by calendar year and property type. To test whether the parameters varied during the sample period, we also aggregated the residuals across calendar years and property types respectively, and estimated stable parameters for the aggregated sets. These results are tabulated in Exhibit 6 and are displayed graphically together with one and two standard deviation error bands in Exhibits 7 to 10 for the parameters α, β, and c (δ is irrelevant because the location parameter is an estimator for the mean and we adjusted for the effect of varying means).

In the case of characteristic exponents α_t estimated by calendar year and property type, 70% (36 of 52) were distinct statistically from 2.0—the characteristic exponent of the normal distribution—with 95% confidence and 63% (33 of 52) were distinct from 2.0 with 99% confidence. In the case of residuals aggregated across property type (the first panel of Table 6), all thirteen sample characteristic exponents α_t were distinct from 2.0 with 99% confidence.

In the case of the skewness parameter β_t for all residuals aggregated across property type, 69% (9 of 13) were statistically significant (i.e., non-zero) with 99% confidence, and one remaining sample value was significant with 95% confidence.

Exhibit 7 displays the sample characteristic exponents α_t of both the aggregated and individual property type residuals. Despite the year-by-year volatility in sample α_t values, after allowing for the width of the error brackets it appears possible in every case that α_t could be time-invariant. Exhibit 8 suggests that α_t is also constant across property type.

By contrast, Exhibit 9 shows clearly that β_t is not time-invariant. Indeed, β_t for all properties displayed a fairly steady decline throughout the test period (the results are indeterminate for individual property types due to the large widths of the error bands). Although β_t displayed both positive and negative values, β_t was negative throughout the entire test period with the exception of the years 1980 and 1981.

Exhibit 10 shows clearly that the scale parameter c is not time-invariant either in the aggregate or by property type. Since c is the stable infinite-variance measure of risk, this means that asset-specific risk is heteroscedastic.[26]

Although Exhibits 7 and 8 suggested to us that α_t was time-invariant during the test period, we must test this proposition rigorously. Similarly, although Exhibit 9 suggested that β_t was not time-invariant during the test period, that too must be tested.

Since all thirteen sample estimators for α_t are asymptotically normal, the proposition that the true values are all equal (i.e., that α_t is time-invariant) can be tested by using the fact that, when it is true,

$$\sum w_i \left(x_i - \bar{x}\right)^2$$

is distributed as χ^2 on twelve degrees of freedom, where each weight w_i is given by the reciprocal of the asymptotic variance of x_i, and $\bar{x}$ is the weighted average of x_i (weighted by the w_i).[27]

26 Mandelbrot (1963a) made the same observation in the case of commodity price changes.

The last column of Exhibit 11 shows the year-by-year χ^2 components for the sample characteristic exponents with the total for the thirteen-year period at the bottom of the column. The total is 21.40, which is marginally larger than the 0.05 significance level of 21.03 for twelve degrees of freedom, and smaller than the 0.01 significance level of 26.22. However, an examination of the year-by-year components of χ^2 reveals that 40% of the total comes from just one year—1991. We note that 1991 was an exceptional year for the commercial real estate market; specifically, it was the year in which the market came closest to total gridlock with few transactions. Since a key ingredient in property valuation is referral to recent sales of comparable properties, the paucity of transactions presumably contributed to exceptionally large uncertainty in the valuation process, and consequently also to exceptionally large uncertainty in the shape of the sample distribution of appraisal-based returns for 1991.[28] If 1991 is treated as an outlier caused by exceptional market conditions, it can be inferred from the remaining components of χ^2 that the time-invariance of α_t under ordinary market conditions during the test period was consistent with the observed data.

To test whether the 1991 data distorted our estimate of α, we reestimated α for the entire sample period using all returns except those from 1991. The reestimated value for α was 1.466—well within the 95% confidence interval around the α value estimated from the returns for all thirteen years of the sample period. The χ^2 test of the hypothesis that α was constant during the twelve-year modified sample period (1980-90 and 1992) yielded a χ^2 value of 12.39, which is well below the 0.05 significance level of 19.68 for eleven degrees of freedom. In fact, this χ^2 value is less than the 0.25 significance level of 13.70 for eleven degrees of freedom, and is only slightly larger than the 0.50 significance level of 10.34 for eleven degrees of freedom. Thus, the result of the χ^2 test confirms the hypothesis that the annual value of α was constant throughout the modified sample period at the 0.25 significance level.

Since the deletion of the 1992 returns from the data set produces an estimate for α that differs insignificantly (in the statistical sense) from the estimate of α based on the full set of returns from the thirteen-year sample period, we will continue to rely on the latter value (Exhibit 6, All Properties Combined) and the 95% confidence interval generated by the McCulloch error estimates—1.477 ±0.038—as the best estimate of α available based on the Russell-NCREIF individual property returns for the years 1980-92.

The χ^2 test can also be used to test whether, for each year during the sample period, the individual property type α estimates are consistent with the hypothesis that the true values of α for the various property types are identical. More precisely, for each year in the sample period, let P_t be the hypothesis that the true values of α for the four property types in year *t* are identical (note that this does not assume that the true value for α is time-invariant). By computing the weighted average of sample property type α's for each year, the analog of the χ^2 test described above can be applied to test hypothesis P_t. This time the critical χ^2 value is 7.81, i.e., the 0.05 significance level of the χ^2 function for three degrees of freedom.

27 Irwin (1942) and James (1951) presented detailed developments of this test in the respective cases of independently distributed normal and asymptotically normal variables.

28 Giliberto (1992) presented evidence that appraisal-based property returns were biased downward relative to equity real estate investment trust returns from 1987 through mid-1992 (which was as far as his data extended), and that this bias was most pronounced during the years 1990-91. He attributed the bias to a paucity of actual property sales during the period, observing that sales provide market signposts needed by appraisers to enable them to determine an appropriate (market-based) capitalization rate.

The resulting thirteen χ^2 values are shown in the next-to-last column of Exhibit 11 (the corresponding χ^2 for the data aggregated across the sample period is shown at the bottom of the column). In every case, the observed sample value is not only below the 0.05 significance level, but is also below the 0.10 significance level. Thus, the observed values are consistent with the conclusion that all of the thirteen hypotheses P_t are correct. This is very strong empirical support for the compound hypothesis that all P_t are correct, since even if the compound hypothesis is correct there is only a 25.42% chance that the χ^2 tests would confirm all thirteen hypotheses at the 0.10 level.[29]

The analogous χ^2 test for β_t can be used to test the proposition that β_t was time-invariant during the test period. The last column of Exhibit 12 shows the year-by-year χ^2 components of the skewness parameter with the total for the thirteen-year period at the bottom of the column. The total is 149.66, which is enormously larger than the 0.005 significance level of 28.30 for twelve degrees of freedom. Thus, there is no reasonable possibility that β_t was time-invariant during the sample period.

The above analysis implies both that (1) real estate investment risk during the sample period was heteroscedastic, and (2) during virtually all sample subperiods and across property type, stable infinite-variance skewed asset-specific risk functions with a characteristic exponent of approximately 1.477 modeled the observed distributions of return residuals better than normally distributed risk candidates.

We point out that a finite-variance model has been used in other studies to account for the fat-tailed appearance of investment returns and financial pricing data. This alternative model—known as the "mixture-of-normals" model—is based on the observation that it is possible to mimic the fat-tailed appearance of sample sets from an infinite-variance distribution by mixing samples from several normal distributions with different standard deviations. However, it is exceedingly difficult to imagine an economic process that could mix samples from different normal distributions in such a way as to generate nearly fifty distinct sample distributions across which skewness and scale parameters vary substantially but which have statistically identical characteristic exponents.

While we have not compared stable infinite-variance risk models with all other possible finite-variance risk models, micromarket considerations described in Section 4 suggest that, absent definitive empirical evidence supporting the superior fit of non-stable candidates, stable distributional risk models are preferable to non-stable risk alternatives on theoretical economic grounds. [30]

Implications for Portfolio Management

In the era of Modern Portfolio Theory, the central task of portfolio management is considered to be the optimization of the portfolio return/risk trade-off, subject to portfolio constraints created by investment policy. This involves asset selection and allocation to achieve two independent

29 This is an application of the binomial distribution—specifically, the probability of thirteen independent successes in thirteen attempts when the probability of each individual success is 90%. The corresponding probability of thirteen independent successes in thirteen attempts is 51.33% when the probability of each individual success is assumed to be 95%.

30 Mandelbrot (1963b) and Fama (1963) op. cit., cf. Blattberg and Gonedes (1974).

objectives: (1) minimization of the combined effect of asset-specific risk, and (2) optimization of the trade-off between portfolio return and systematic/sector risk.

The approach to this problem taken by virtually all portfolio research is: (a) specify the largest tolerable combined asset-specific risk; (b) calculate the minimum number of assets necessary to ensure that the combined effect of asset-specific risk is below the critical threshold; and (c) solve problem (2) under the additional constraint that investment funds be diversified among at least the number of assets determined in (b). In particular, this solution is taken in MPT approaches to the management of U.S. stock portfolios.

To see what is involved in satisfying the additional constraint imposed by (b), it is instructive to make the following simplifying assumptions: all asset-specific risk functions are stable with the same characteristic exponent α and have the same skewness parameter β, all individual assets have the same level of asset-specific risk (proxied by the scale parameter c of the distribution for the common asset-specific risk function), and the same percentage of the total portfolio value is invested in each component asset in the optimal portfolio. Then, letting p represent the portfolio, f the common asset-specific risk function, and using the relation between scale parameters of sums of stable random variables described in Equation (2):[31]

$$\begin{aligned} c_p^{\alpha} &= c_{(1/n)f_1}^{\alpha} + \ldots + c_{(1/n)f_n}^{\alpha} \\ &= nc_{(1/n)f}^{\alpha} = n(1/n)c_f^{\alpha} \\ &= n(1/n)^{\alpha} c_f^{\alpha} = n^{(1-\alpha)} c_f^{\alpha} \end{aligned}$$

This implies that:

$$c_p = n^{(1/\alpha)-1} c_f \qquad (5)$$

Exhibit 13 shows the impact of varying α upon reduction in asset-specific risk for various numbers of properties in a portfolio. For any given α>1.0, the reduction in asset-specific risk increases with increasing n. As α diminishes to 1.0 from its upper limit of 2.0, the reduction in asset-specific risk likewise diminishes for any given n> 1.

The sample value α=1.477 from the preceding section implies the following practical estimate for the effect of portfolio diversification on asset-specific risk reduction:

$$c_p \approx n^{-0.323} c_f \qquad (5')$$

A typical closed-end real estate fund has 10 to 15 properties. Under the above assumptions, the magnitude of combined asset-specific risk for such a fund is between 42% and 48% of the magnitude of asset-specific risk for a single property portfolio. However, if the asset-specific risk were normally distributed, the combined asset-specific risk would be between 26% and 32%.

Alternatively, if the question of risk reduction is rephrased to ask the number of assets n_k needed in a portfolio to achieve a reduction of asset-specific risk by a specified factor of k, then the answer is as follows: n_k is the smallest integer at least as large as k raised to the power $1/0.323$. In mathematical notation,

$$n_k = k^{\alpha/(\alpha-1)} + 1 \approx k^{3.10} + 1 \qquad (6)$$

This implies that the number of properties in a portfolio needed to achieve a four-fold reduction in the magnitude of combined asset-specific risk is 74—compared with only 16 properties if asset-specific risk were normally distributed. Similarly, the number of properties in a portfolio needed to achieve a ten-fold reduction in combined asset-specific risk is

31 Cf. Fama and Miller (1972), pp. 268-270, and Fama (1965b).

1,259—compared with 100 properties if asset-specific risk were normally distributed. In other words, if purchases are restricted to institutional-grade properties, equally weighted investments in two-thirds of the properties currently in the Russell-NCREIF data base would be needed to achieve a ten-fold reduction in the magnitude of combined asset-specific risk.

The effect of varying α upon the portfolio size needed to achieve risk reduction by various specified factors k is shown in Exhibit 14.

Conclusions

The analysis in this study supports the unequivocal conclusion that individual (continuously compounded) annual property returns in the Russell-NCREIF combined data base are not normally distributed for any calendar year during the period 1980-92. It also supports the conclusion that, for each calendar year t in that interval, there is a stable infinite-variance distribution with characteristic exponent α_t such that the return on each property for year t can be represented as the average (mean) return for that year on properties of the same commercial type plus a random sample from the stable distribution for that year, and furthermore that these samples are independent for distinct properties or calendar years. These stable distributions can be considered to represent real estate asset-specific risk.

Our data analysis strongly implies that both the skewness and magnitude of real estate asset-specific risk change over time, i.e., real estate risk is heteroscedastic with respect to both the amount of risk and the shape of the risk distribution. However, the analysis also supports the conclusion that there is a single value for the characteristic exponent of asset-specific risk across both calendar year and property type. A statistical estimate of this common value for the characteristic exponent α together with a 95% confidence interval around this value is 1.477 ±0.038, based on a sample distribution of 13,958 annual property returns over the 13-year sample period. This interval is so far from 2.0—the value for a normal distribution—that it has profound implications for real estate portfolio management.

The low observed value for the characteristic exponent implies that reduction of asset-specific investment risk to levels readily achievable in the stock and bond markets through asset diversification requires a portfolio of far more real estate assets than would be needed for the case of normally distributed risk. In real estate portfolios subject to institutional quality standards, the appropriate degree of risk reduction across multiple risk factors (locational, economic, etc.) could only be achieved by purchasing most of the institutional-grade properties in the U.S.—a practical impossibility. This implies that institutional real estate portfolio management must be concerned with the asset-specific risk component of each property included in the portfolio as well as with market/systematic and market-sector risk components. Furthermore, the stationarity of the characteristic exponent for investment risk across time and property type is independent of whether or not regional groupings, for instance, provide a meaningful additional risk dimension as some researchers have suggested.

The fact that real estate investment risk has infinite variance also implies that there is no way to measure codependence among property risk functions with the statistical tools currently available. In particular, sample correlations used in multi-factor MPT real estate risk models are fictitious products of flawed data analysis methodology, and do not measure true risk codependence.

This is not to assert that MPT is inapplicable in the real estate context, only that: (1) the current conceptual version of MPT that has been appropriated without modification from stock market analysis is inapplicable, and (2) the asset-specific risk of individual properties must be considered in addition to overall market and sector risk exposures in analyzing relevant portfolio risk.

The observed negative skewness of real estate returns for most years of the sample period and across property type also has a conceptual economic interpretation. As described in Graff and Cashdan (1990), a significant percentage of total real estate return consists of the return on a fixed-income portfolio—i.e., the leases in place. In the case of unleveraged real estate, this fixed-income component acts as an anchor on capital appreciation, making it difficult to achieve dramatic upside returns except during years when major incremental tax breaks favoring real estate versus other assets are anticipated or enacted. This effect shows up in a study of returns as a truncation of the upper end of the returns distribution, which in quantitative descriptions of the distribution translates into negative skewness (cf. the sample values for β in Exhibit 6).

A final observation concerns the accuracy of appraisal-based returns data relative to transaction-based data. In our view, the fact that thousands of appraisals (or valuations) by real estate professionals across the country over a thirteen-year period form sample distributions with statistically indistinguishable characteristic exponents across calendar years and across property types suggests strongly that the real estate community has a common perception of asset value and the sources of that value that has remained constant across changing market regimes of liquidity, tax benefits, credit access, and supply and demand of product. On the other hand, the existence of a marginal outlier for the year in which the commercial real estate market came closest to total gridlock—1991—also indicates that most real estate professionals require a moderate number of actual transactions to provide a benchmark for their common perceptions of value.

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Exhibit 1a
Distribution of Log Annual Total Return Residuals
Russell-NCREIF Combined Property Data Bases
All Properties, 1980 to 1992

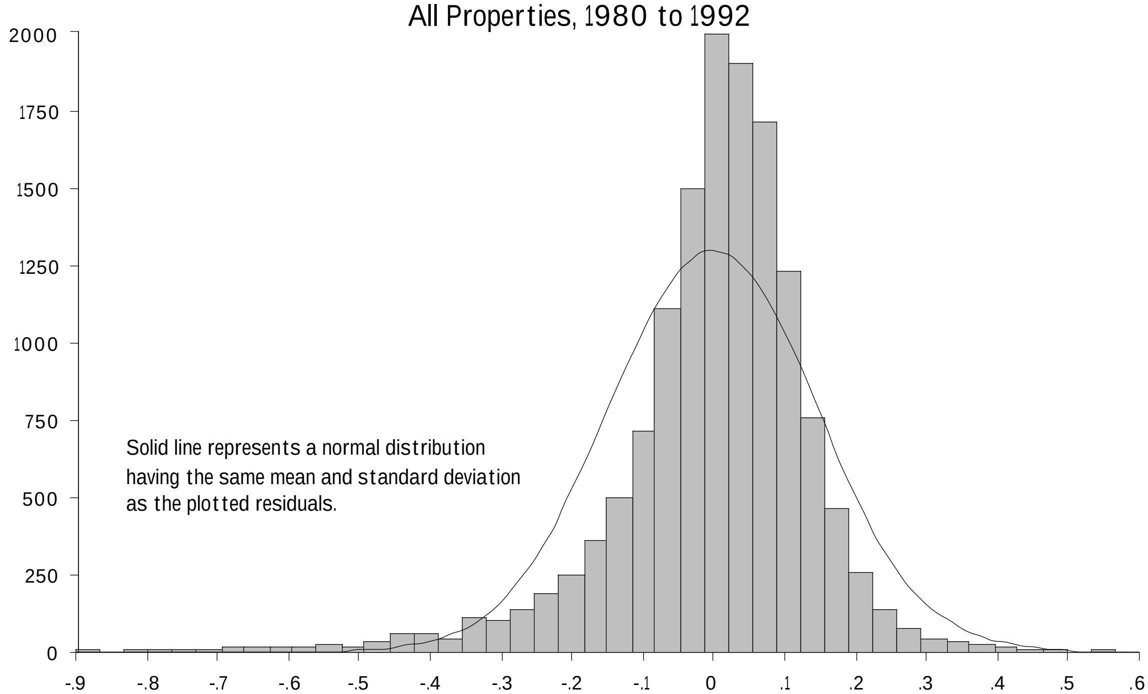


Exhibit 1b
Difference in Frequency, Log Annual Total Return Residuals to Normal Distribution
Russell-NCREIF Combined Property Data Bases
All Properties, 1980 to 1992

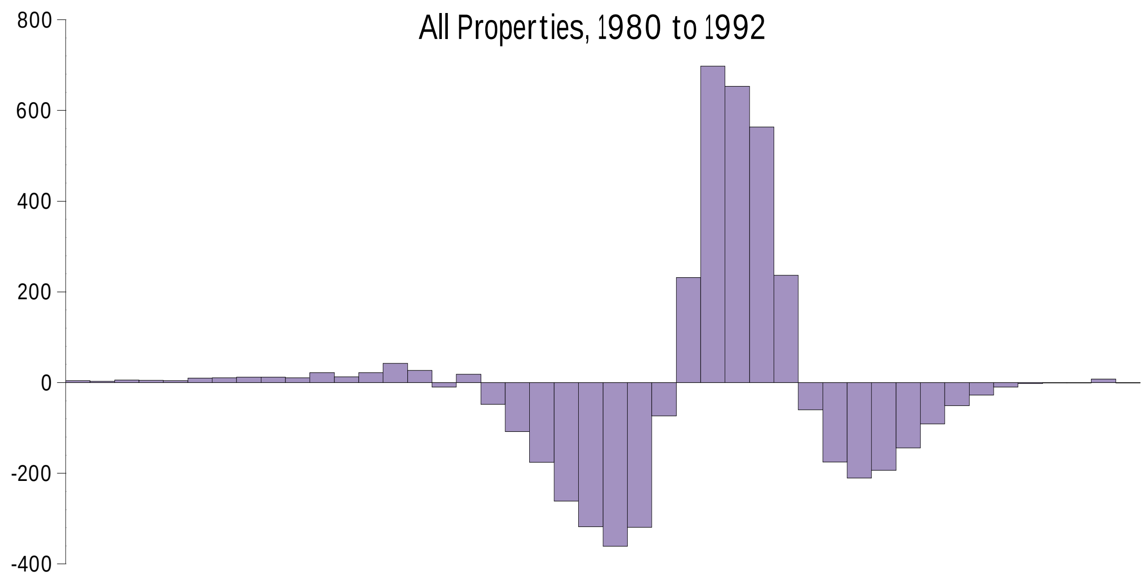


Exhibit 2a
Distribution of Log Annual Total Return Residuals
Russell-NCREIF Combined Property Data Bases
Office Properties, 1980 to 1992

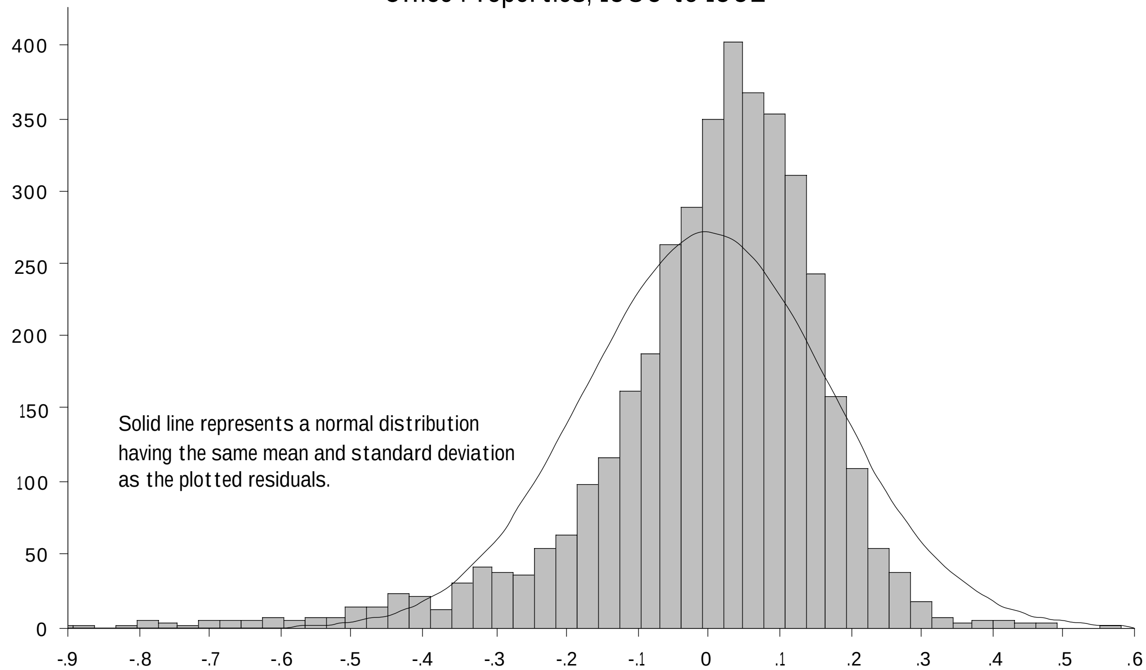


Exhibit 2b
Difference in Frequency, Log Annual Total Return Residuals to Normal Distribution
Russell-NCREIF Combined Property Data Bases
Office Properties, 1980 to 1992

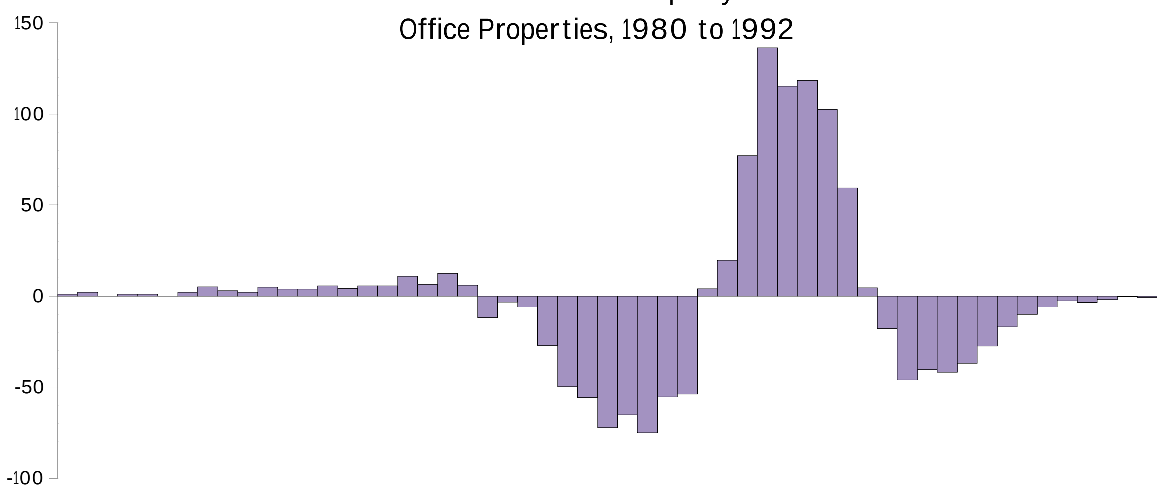


Exhibit 3a
Distribution of Log Annual Total Return Residuals
Russell-NCREIF Combined Property Data Bases
Retail Properties, 1980 to 1992

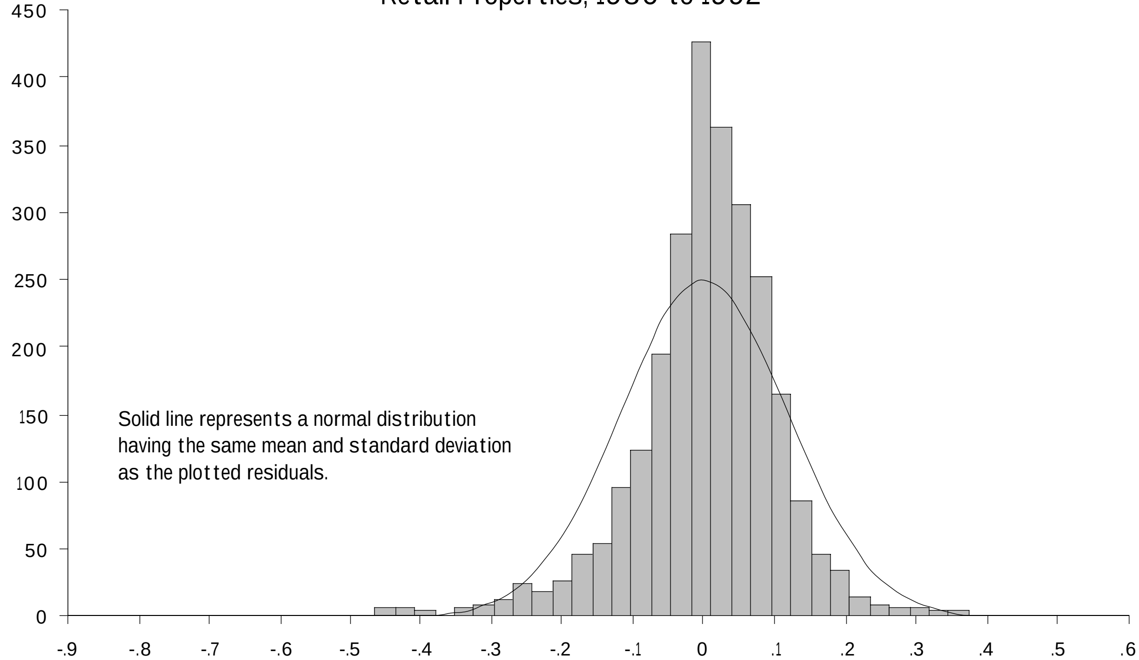


Exhibit 3b
Difference in Frequency, Log Annual Total Return Residuals to Normal Distribution
Russell-NCREIF Combined Property Data Bases
Retail Properties, 1980 to 1992

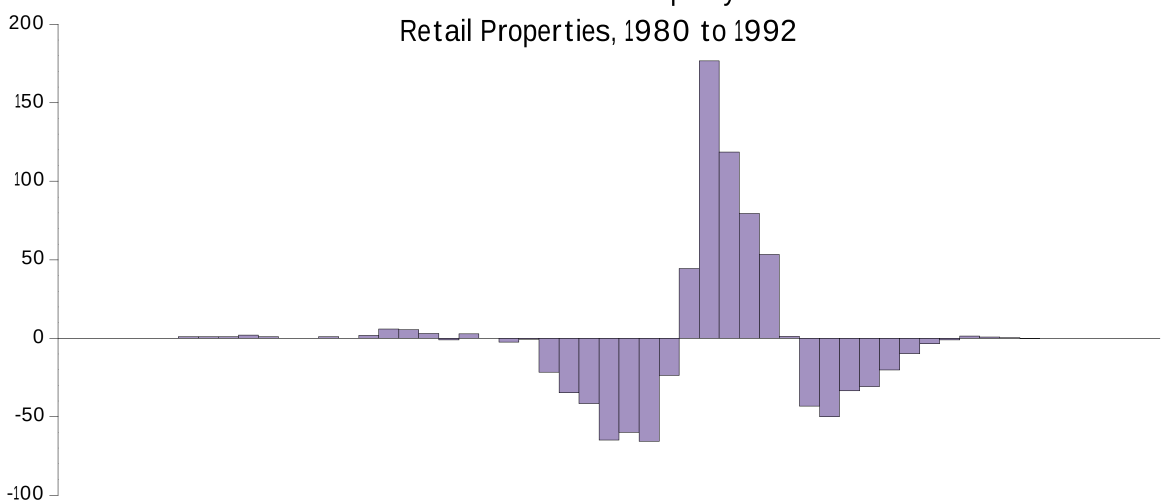


Exhibit 4a

Distribution of Log Annual Total Return Residuals

Russell-NCREIF Combined Property Data Bases

Warehouse Properties, 1980 to 1992

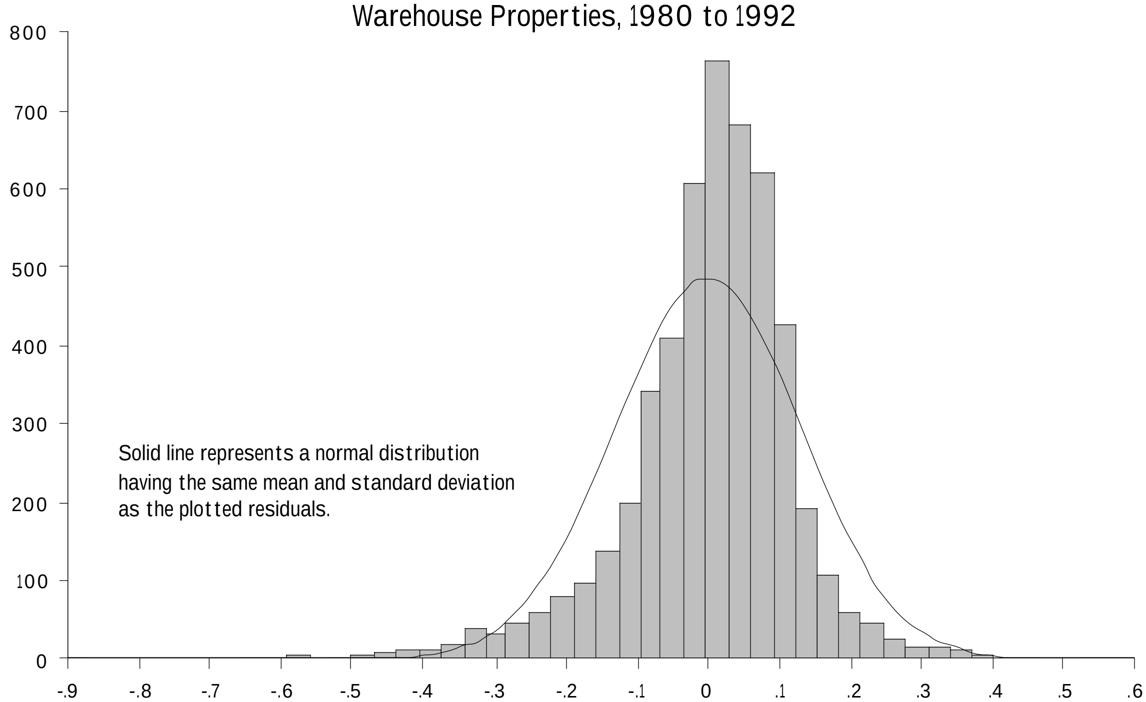


Exhibit 4b

Difference in Frequency, Log Annual Total Return Residuals to Normal Distribution

Russell-NCREIF Combined Property Data Bases

Warehouse Properties, 1980 to 1992

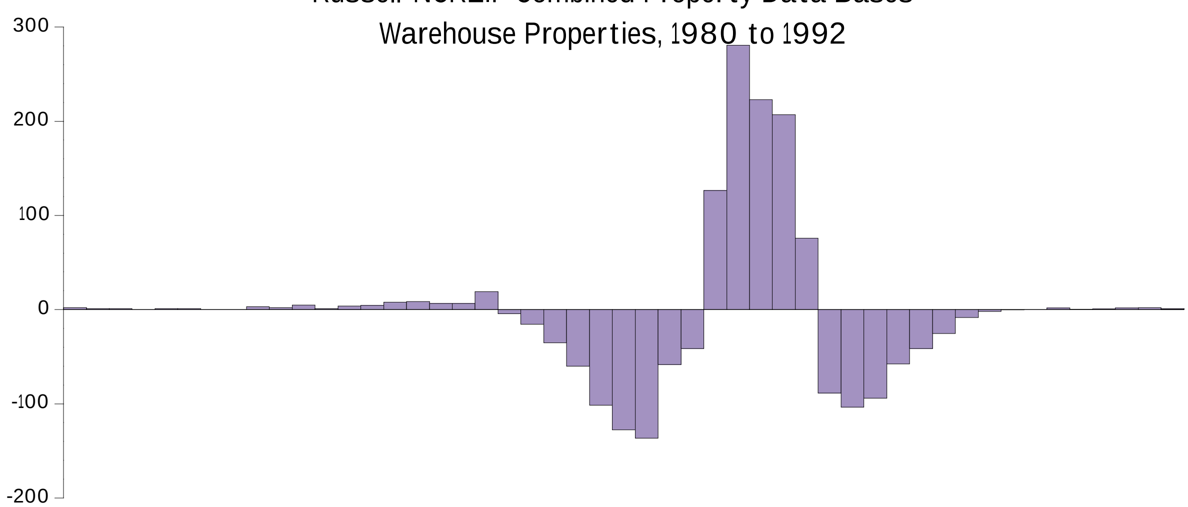


Exhibit 5a
Distribution of Log Annual Total Return Residuals
Russell-NCREIF Combined Property Data Bases
Research & Development Properties, 1980 to 1992

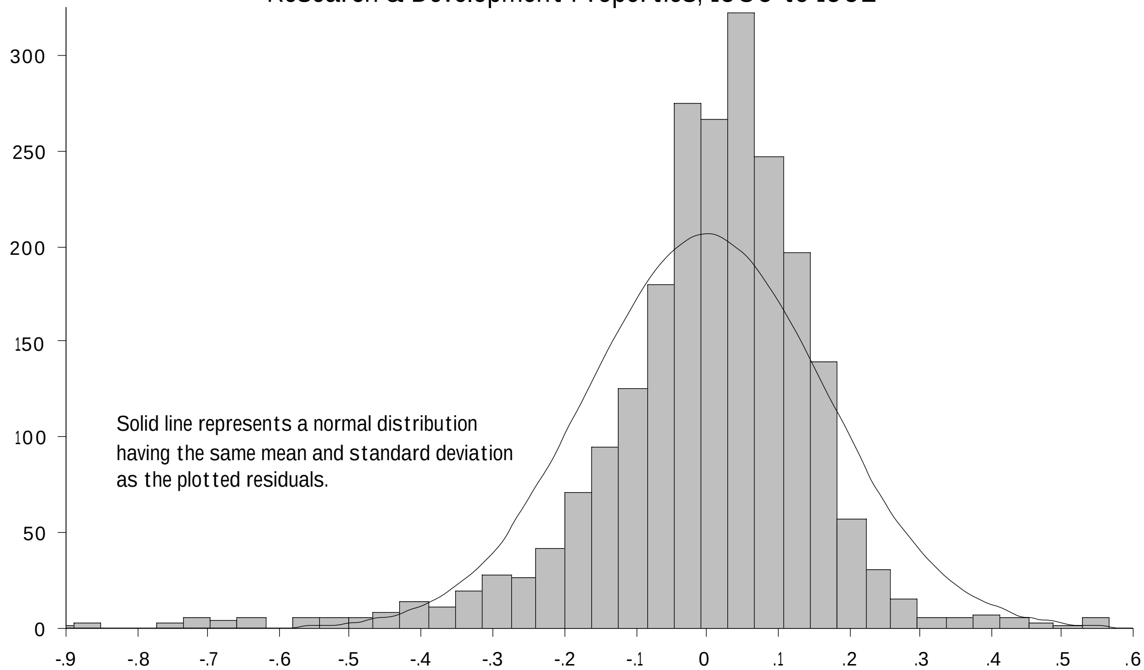


Exhibit 5b
Difference in Frequency, Log Annual Total Return Residuals to Normal Distribution
Russell-NCREIF Combined Property Data Bases
Research & Development Properties, 1980 to 1992

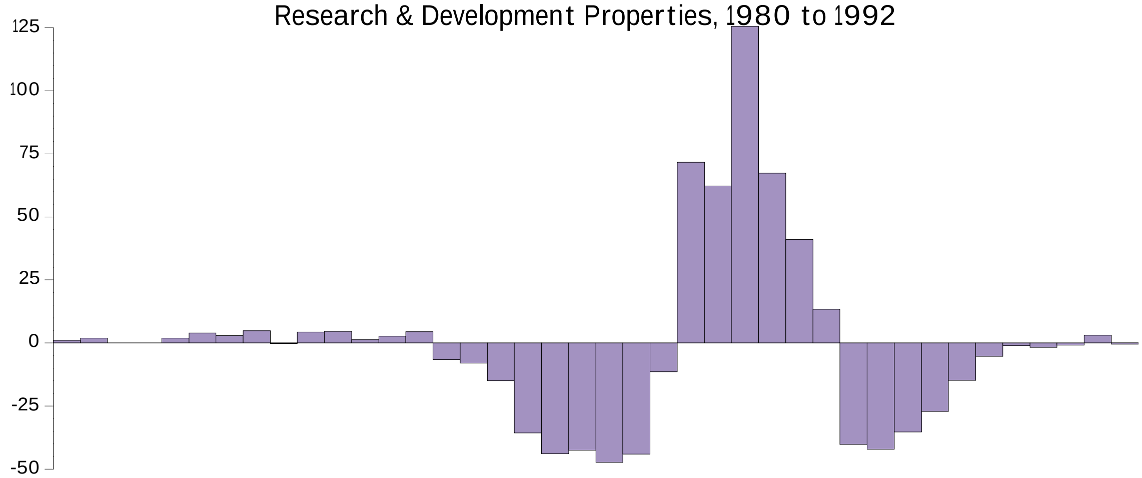


Exhibit 6
Stable Distribution Parameters for Russell-NCREIF Combined Property Data Base
Log Annual Total Return Residuals & Mean Returns & Number of Properties

All Properties Combined:

Year or Period	α		β		c	Mean Return	Number of Properties
1992	1.526	**	-1.000	**	0.082	-0.044	1,715
1991	1.631	**	-1.000	**	0.089	-0.065	1,637
1990	1.348	**	-0.797	**	0.059	-0.011	1,451
1989	1.329	**	-0.536	**	0.059	0.036	1,299
1988	1.489	**	-0.543	**	0.064	0.053	1,248
1987	1.405	**	-0.494	**	0.071	0.038	1,158
1986	1.462	**	-0.465	**	0.057	0.062	1,098
1985	1.425	**	-0.207	*	0.050	0.100	972
1984	1.374	**	-0.001		0.049	0.116	903
1983	1.376	**	-0.194		0.056	0.105	886
1982	1.371	**	-0.039		0.052	0.087	692
1981	1.233	**	0.459	**	0.059	0.160	507
1980	1.472	**	1.000	**	0.054	0.155	392
1980-92	1.477	**	-0.466	**	0.068	0.038	13,958

Office Properties:

Year or Period	α		β		c	Mean Return	Number of Properties
1992	1.656	*	-1.000		0.109	-0.106	463
1991	2.000		-1.000		0.137	-0.149	455
1990	1.455	**	-1.000	**	0.084	-0.074	421
1989	1.481	**	-1.000	**	0.080	-0.014	400
1988	1.543	**	-1.000	*	0.072	-0.002	380
1987	1.302	**	-1.000	**	0.080	-0.026	363
1986	1.411	**	-0.884	**	0.066	0.018	348
1985	1.437	**	-0.262		0.058	0.072	299
1984	1.312	**	-0.070		0.049	0.101	255
1983	1.277	**	-0.270		0.052	0.100	239
1982	1.749		1.000		0.062	0.102	175
1981	1.478	*	0.660		0.063	0.173	92
1980	2.000		1.000		0.063	0.153	65
1980-92	1.517	**	-0.783	**	0.084	-0.011	3,955

Exhibit 6 (continued)
Stable Distribution Parameters for Russell-NCREIF Combined Property Data Base
Log Annual Total Return Residuals & Mean Returns & Number of Properties

Retail Properties:

Year or Period	α		β		c	Mean Return	Number of Properties
1992	1.970		-1.000		0.073	0.002	392
1991	1.643	*	-1.000		0.069	-0.022	378
1990	1.322	**	-0.610	**	0.035	0.049	284
1989	1.093	**	-0.249		0.035	0.074	221
1988	1.534	**	-0.003		0.058	0.106	206
1987	1.351	**	-0.201		0.048	0.100	195
1986	1.423	**	0.196		0.043	0.111	186
1985	1.268	**	0.003		0.038	0.118	171
1984	1.706		0.957		0.043	0.130	166
1983	1.366	**	-0.072		0.044	0.115	162
1982	1.222	**	-0.072		0.044	0.081	119
1981	1.669		-0.637		0.056	0.090	98
1980	1.522		0.549		0.045	0.128	74
1980-92	1.545	**	-0.400	**	0.055	0.065	2,652

Warehouse Properties:

Year or Period	α		β		c	Mean Return	Number of Properties
1992	1.512		-1.000		0.070	-0.013	616
1991	1.826		-1.000		0.082	-0.024	560
1990	1.400	**	-1.000	**	0.053	0.012	496
1989	1.306	**	-0.375	**	0.051	0.063	427
1988	1.434	**	-0.288		0.059	0.082	426
1987	1.371	**	-0.371	*	0.059	0.076	384
1986	1.390	**	-0.162		0.047	0.089	376
1985	1.568	**	-0.181		0.049	0.116	343
1984	1.263	**	-0.104		0.041	0.109	356
1983	1.458	**	-0.424	*	0.059	0.092	370
1982	1.226	**	-0.203		0.049	0.077	308
1981	1.304	**	0.499	**	0.051	0.150	247
1980	1.250	**	0.284		0.057	0.162	205
1980-92	1.500	**	-0.510	**	0.061	0.062	5,114

Exhibit 6 (continued)
Stable Distribution Parameters for Russell-NCREIF Combined Property Data Base
Log Annual Total Return Residuals & Mean Returns & Number of Properties

Research & Development Properties:

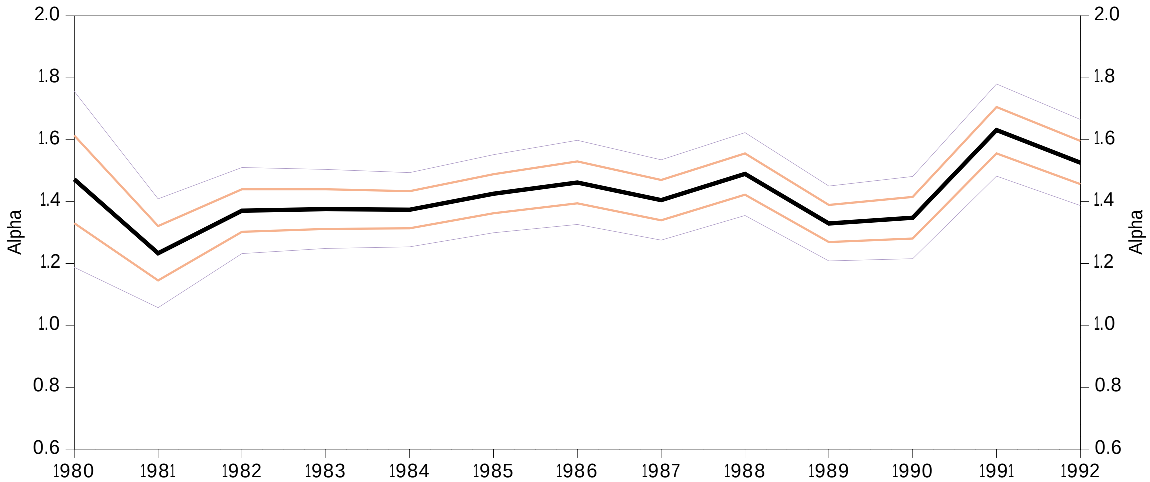
Year or Period	α		β		c	Mean Return	Number of Properties
1992	2.000		-1.000		0.124	-0.075	244
1991	1.852		-1.000		0.114	-0.068	244
1990	1.371	**	-1.000	**	0.066	-0.021	250
1989	1.296	**	-0.433	*	0.054	0.037	251
1988	1.168	**	-0.563	**	0.052	0.041	236
1987	1.449	**	-0.839	*	0.081	0.021	216
1986	1.783		-1.000		0.076	0.037	188
1985	1.235	**	-0.019		0.042	0.101	159
1984	1.715		0.732		0.062	0.150	126
1983	1.361	**	0.593	*	0.053	0.147	115
1982	1.587		0.226		0.051	0.099	90
1981	1.965		1.000		0.140	0.277	70
1980	1.620		1.000		0.073	0.169	48
1980-92	1.531	**	-0.639	**	0.077	0.035	2,237

Statistically significant confidence of non-normality ($\alpha \neq 2.0$) or skewness ($\beta \neq 0$):
** = 99% confidence
* = 95% confidence

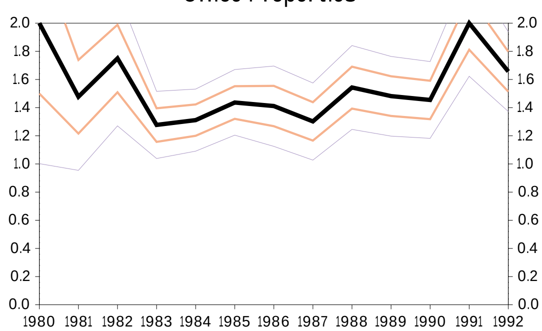
α is the characteristic exponent, and only equals 2.0 for the normal distribution
β is the skewness parameter in the range -1.0 to +1.0
c is the (positive) scale parameter which measures the spread of the distribution about δ

Exhibit 7
Characteristic Exponent "Alpha" of Distributions of Log Annual Total Return Residuals
Russell-NCREIF Combined Property Data Bases, All Properties

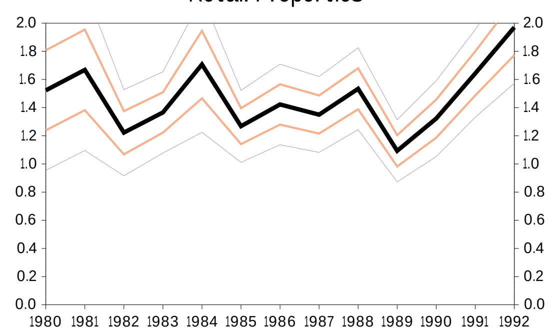
(bands indicate plus and minus one and two standard deviations)



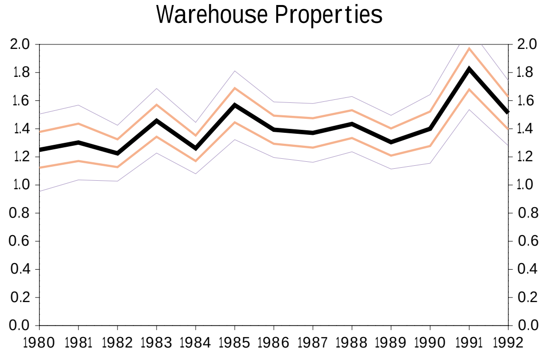
Office Properties



Retail Properties



Warehouse Properties



Research & Development Properties

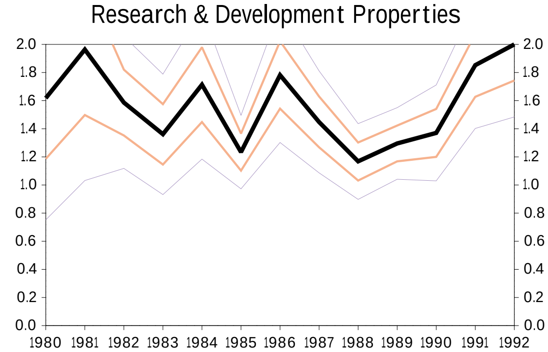


Exhibit 8

Characteristic Exponent "Alpha" of Distributions of Log Annual Total Return Residuals
Four Property Types Total and Individually for the 1980 to 1992 Period
(bands indicate plus and minus one and two standard deviations)

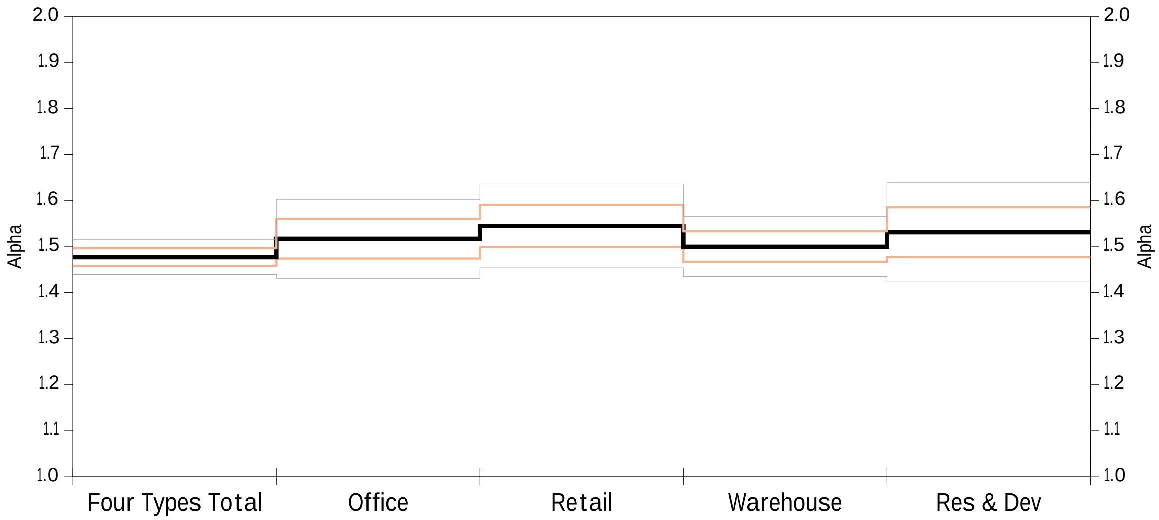


Exhibit 9
Skewness Parameter "Beta" of Distributions of Log Annual Total Return Residuals
Russell-NCREIF Combined Property Data Bases, All Properties
(bands indicate plus and minus one and two standard deviations)

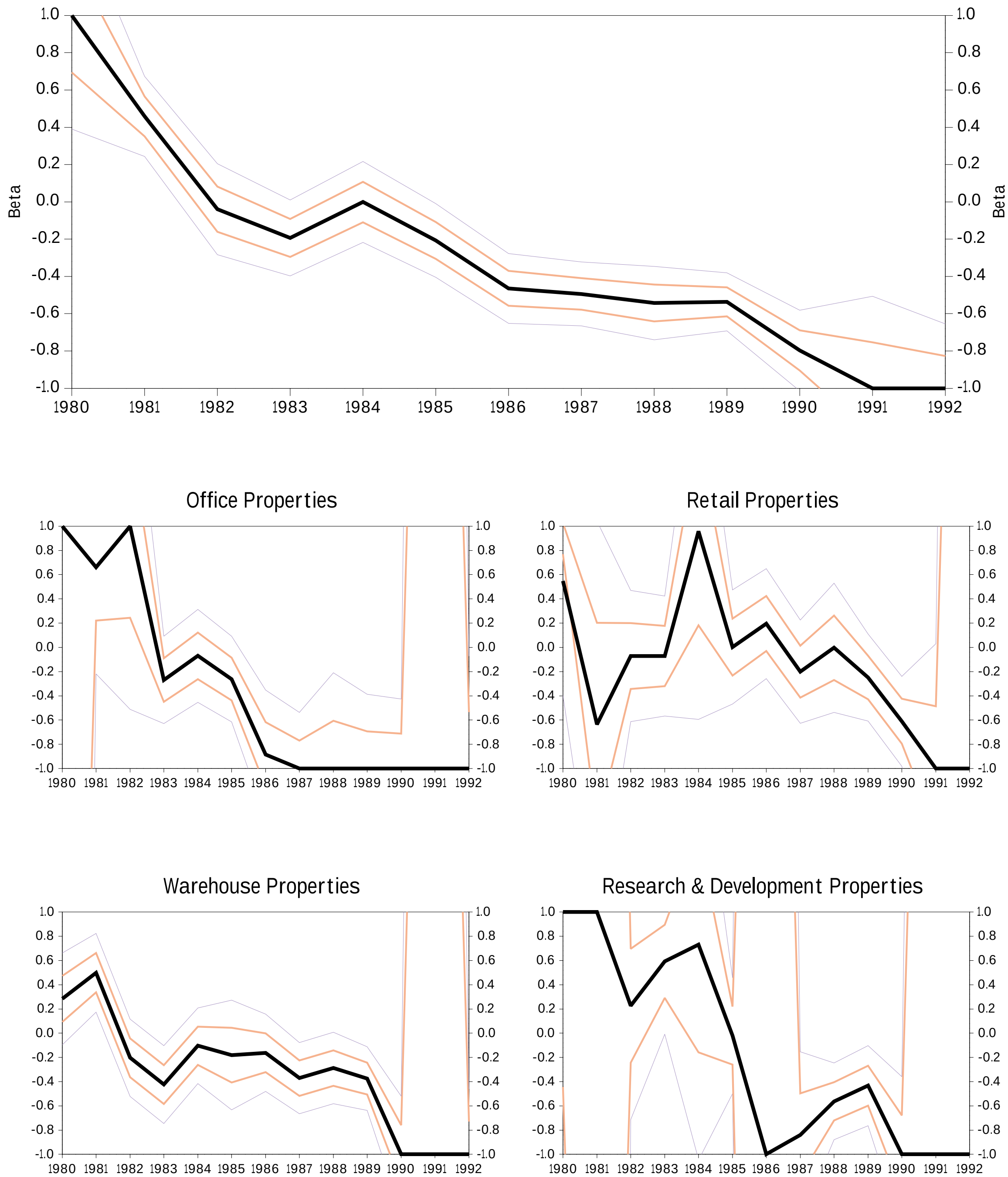
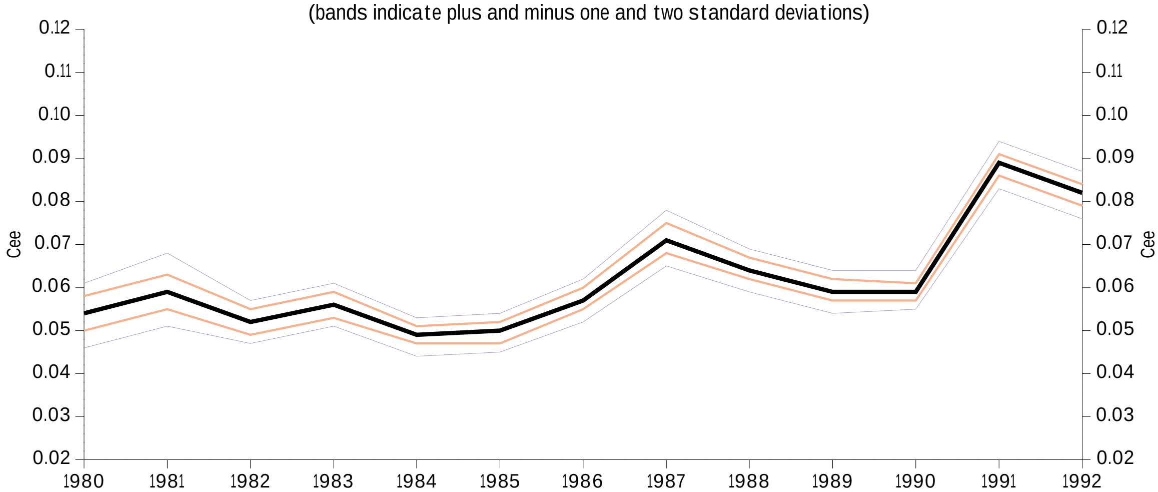
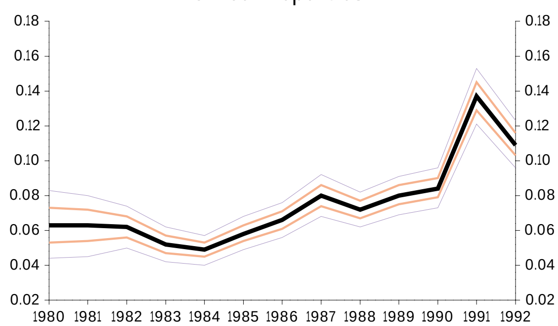


Exhibit 10
Scale Parameter "C" of Distributions of Log Annual Total Return Residuals
Russell-NCREIF Combined Property Data Bases, All Properties

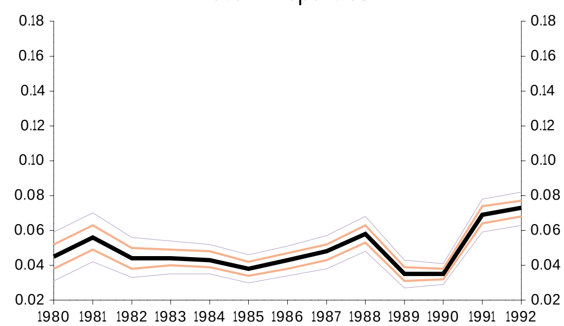
(bands indicate plus and minus one and two standard deviations)



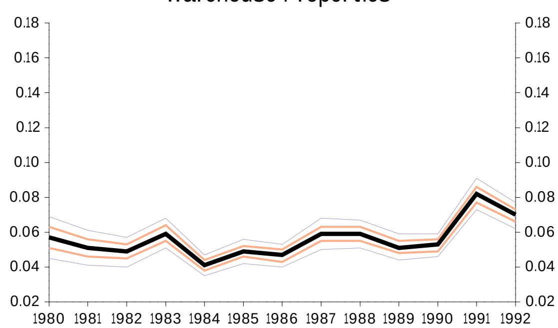
Office Properties



Retail Properties



Warehouse Properties



Research & Development Properties

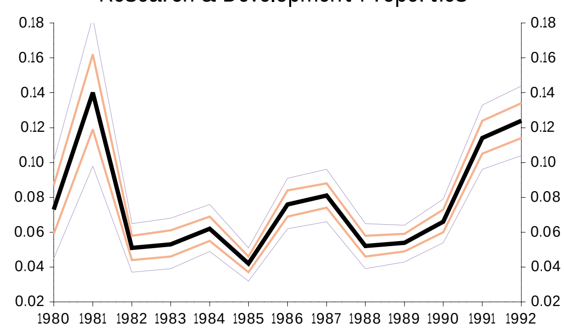


Exhibit 11
Characteristic Exponent α for Russell-NCREIF Combined Property Data Base
Log Annual Total Return Residual Distributions
All Properties, Properties by Type, and Chi-square Goodness of Fit Results

Year or Period	All Properties	Office	Retail	Warehouse	Research & Dev	Annual Property-Type χ^2	Annual Components of Sample Period χ^2
1992	1.526	1.656	1.970	1.512	2.000	5.82	2.56
1991	1.631	2.000	1.643	1.826	1.852	2.21	8.41
1990	1.348	1.455	1.322	1.400	1.371	0.50	1.00
1989	1.329	1.481	1.093	1.306	1.296	4.97	2.01
1988	1.489	1.543	1.534	1.434	1.168	4.81	1.23
1987	1.405	1.302	1.351	1.371	1.449	0.44	0.02
1986	1.462	1.411	1.423	1.394	1.783	2.32	0.49
1985	1.425	1.437	1.268	1.568	1.235	4.60	0.03
1984	1.374	1.312	1.706	1.263	1.715	5.10	0.46
1983	1.376	1.277	1.366	1.458	1.361	1.21	0.37
1982	1.371	1.749	1.222	1.226	1.587	5.68	0.40
1981	1.233	1.478	1.669	1.304	1.965	2.93	4.27
1980	1.472	2.000	1.522	1.250	1.620	3.07	0.16
1980-92	1.477	1.517	1.545	1.500	1.531	2.18	
1980-92 χ^2		18.74	24.84	19.96	21.97		21.40

Exhibit 12
Skewness Parameter β for Russell-NCREIF Combined Property Data Base
Log Annual Total Return Residual Distributions
All Properties, Properties by Type, and Chi-square Goodness of Fit Results

Year or Period	All Properties	Office	Retail	Warehouse	Research & Dev	Annual Property-Type χ^2	Annual Components of Sample Period χ^2
1992	-1.000	-1.000	-1.000	-1.000	-1.000	0.00	14.62
1991	-1.000	-1.000	-1.000	-1.000	-1.000	0.00	7.13
1990	-0.797	-1.000	-0.610	-1.000	-1.000	2.54	17.87
1989	-0.536	-1.000	-0.249	-0.375	-0.433	4.60	6.31
1988	-0.543	-1.000	-0.003	-0.288	-0.563	6.16	4.25
1987	-0.494	-1.000	-0.201	-0.371	-0.839	8.39	3.25
1986	-0.465	-0.884	0.196	-0.162	-1.000	9.76	1.79
1985	-0.207	-0.262	0.003	-0.181	-0.019	1.14	1.80
1984	-0.001	-0.070	0.957	-0.104	0.732	2.58	9.82
1983	-0.194	-0.270	-0.072	-0.424	0.593	9.29	2.06
1982	-0.039	1.000	-0.072	-0.203	0.226	3.02	6.12
1981	0.439	0.660	-0.637	0.499	1.000	1.96	55.35
1980	1.000	1.000	0.549	0.284	1.000	0.49	19.29
1980-92	-0.466	-0.783	-0.400	-0.510	-0.639	13.77	
1980-92 χ^2		34.57	16.98	71.85	23.77		149.66

Exhibit 13
Risk Reduction for Various α and Number of Assets

	Number of Assets						
α	1	2	4	8	10	20	100
2.00	1	0.707	0.500	0.354	0.316	0.224	0.100
1.90	1	0.720	0.519	0.373	0.336	0.242	0.113
1.80	1	0.735	0.540	0.397	0.359	0.264	0.129
1.70	1	0.752	0.565	0.425	0.387	0.291	0.150
1.60	1	0.771	0.595	0.459	0.422	0.325	0.178
1.50	1	0.794	0.630	0.500	0.464	0.368	0.215
1.40	1	0.820	0.673	0.552	0.518	0.425	0.268
1.30	1	0.852	0.726	0.619	0.588	0.501	0.346
1.20	1	0.891	0.794	0.707	0.681	0.607	0.464
1.10	1	0.939	0.882	0.828	0.811	0.762	0.658
1.00	1	1.000	1.000	1.000	1.000	1.000	1.000
0.90	1	1.080	1.167	1.260	1.292	1.395	1.668

Exhibit 14
Number of Assets Needed for Risk Reduction by the Factor *k*

	Factor *k*						
α	1	2	4	8	10	20	100
2.00	1	4	16	64	100	400	10,000
1.90	1	5	19	81	130	558	16,682
1.80	1	5	23	108	178	846	31,623
1.70	1	6	29	156	269	1,445	71,969
1.60	1	7	41	256	465	2,948	215,444
1.50	1	8	64	512	1,000	8,000	1,000,000
1.40	1	12	128	1,448	3,163	35,778	10,000,000
1.30	1	21	407	8,192	21,545	434,307	4.6×10^{8}
1.20	1	64	4,096	262,144	1,000,000	6.4×10^{7}	1.0×10^{12}