Real Estate Return Distributions Using Maximum Likelihood Estimation (MLE): New Technology, New Results

by

Michael S. Young

2635 North 162nd Lane
Goodyear, Arizona 85395
phone: 623-523-3855 // e-mail: mikero1@mac.com

and

Roger J. Brown, Ph.D.

Colvin Institute of Real Estate Development
School of Architecture, Planning and Preservation
University of Maryland, College Park, MD 20742
phone: 619-445-3405 // e-mail: rjb21@cox.net

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Abstract: The estimation of parameters of real estate return distributions is affected by the tools used to do the work. Consistent with previous studies, investment risk models with infinite variance describe distributions of individual property returns in the NCREIF database over the period 1980 to 2010. Applying Maximum Likelihood Estimation (MLE) to the historic data shows real estate investment risk to be heteroscedastic, but the Characteristic Exponent of the investment risk function varies more among property types than previously reported. Apartment properties introduced in this study evidence the same basic performance characteristics as Office, Retail, and Industrial properties.

Key words: Asset-specific risk, return distributions, Maximum Likelihood Estimation, non-Normality, diversification, institutional investing

"First, show the world that the airplane can fly. Then, let others refine it."
Benoit Mandelbrot, personal conversation with author

The application of technology to the defining the shape of financial asset return distributions has been a sporatic effort over the past fifty years. With each new attempt has come some improvement in precision and thereby greater and greater appreciation for just how far removed financial asset return distributions are from Gaussian Normal distributions, the default presumption in countless academic studies, educational courses, and practical applications.

In the beginning there was double-log graph paper upon which Benoit Mandelbrot (1963b) plotted cotton prices over a sixty year period using a #2 lead pencil. Regardless of the time interval he chose, the patterns always looked the same and resembled his earlier work on noise in electronic transmissions, namely a symmetrical or cyclical non-periodic pattern with respect to scaling but an erratic pattern with respect to a Normal distribution of price levels. Had the return distributions been Gaussian Normal, the line connecting the points would have been a straight line, but that was not to be. With departures from the straight-line expectation more pronounced at the high positive and low negative ends of the plot, Mandelbrot concluded that the distributions were not likely to be Normal but some other stable distribution with infinite or undefined variance.

In his work with cotton prices and some subsequent work with stock prices, Mandelbrot continued a theme that ran throughout his career in myriad and diverse fields. He routinely turned algebraic problems into geometric problems and solved these intuitively with shapes that exposed the patterns and made solutions and explanations accessible to a lay audience. Mandelbrot's pictures—Cantor sets, Julia sets, and eventually the Mandelbrot sets themselves—all demonstrate some degree of roughness or irregularity inherent in physical objects or mathematical results from physical processes that seem far more universal than mathematically simpler (and more teachable) models like the Gaussian Normal distribution. Mandelbrot's catch-all term for this degree of roughness or irregularity is its "fractal dimension," something other than the familiar integer dimensions of the physical world: one, two, three, or four.

While the fractal view of the world has been shown repeatedly to be a superior representation of the real world, the predominant view espoused in academia and among practioners is that simplified models rooted in easily-taught and easily-comprehended models are "good enough."

Being just "good enough" may suffice for teaching general principles, but it can fall woefully short of providing tools for practical applications of those principles. One case in point is the degree to which adding uncorrelated risky assets to a portfolio might reduce the non-systematic risk of the portfolio. Non-systematic risk reduction at the rate of one over the square root of the number of assets is commonly taught in investment classes and espoused by investment managers. However, research by Young and Graff (1995) has shown that a more accurate picture of commercial property return distributions implies non-systematic risk reduction at the rate of about one over the cube root of the number of assets.

These differences are non-trivial. Where it would take 100 assets whose returns were Normally distributed (i.e., having a Characteristic Exponent of 2.0) to affect a ten-fold reduction in non-systematic risk in a portfolio, it would take 1,000 assets to produce the same result for assets whose returns approximate the empirical results of earlier studies by Young and Graff (1995) and Young (2008) (i.e., having a Characteristic Exponent of 1.5).

In the real world business of real estate investment management, the promise of risk reduction through diversification over a reasonable number of risky assets becomes hollow and unattainable for even the wealthiest of institutional investors.

How the investment world became persuaded to embrace the Gaussian Normal view despite repeated suggestions that the real world was anything but Gaussian Normal is worthy tale.

A Brief History of Modern Financial Analysis

In a distinct break with the investment analytics of the past which included financial statement analysis espoused by Graham and Dodd (1934) and a variety of charting techniques of unknown parentage, Markowitz began tinkering with the present value model of stock prices introduced by Williams (1938). Markowitz found present value without consideration of risk to be a shortcoming of the model and sought to improve upon it. His effort culminated what is certainly the seminal work of Modern Portfolio Theory (MPT), Markowitz (1952), in which risk and return are mathematically independent of one another but positively related. A natural implication of the Markowitz model is that non-systematic or asset-specific risk can be reduced by combining uncorrelated or weakly correlated risky assets of the same class into portfolios. This is the beneficial effect of diversification espoused by investment managers worldwide.

Coincidentially, at the time Mandelbrot was working on cotton prices, the investment world was rediscovering *Theory of Speculation*, the work of Louis Bachelier circa 1900. Bachelier envisioned changes in market prices as following a random walk, which leads to the conclusion that a Gaussian Normal distribution is the operative pattern. Because the Normal distribution is one of the principal foundations of modern statistics, adapting its concepts and techniques to financial asset prices or returns seemed natural and convenient for mathematically-inclined educators and their disciples heading to Wall Street.

After publication by Fisher and Lorie (1964) of monthly returns on New York Stock Exchange-traded securities between 1926 and 1962, empirical investigations of stock price behavior and testing of potential theoretical and practical models began in earnest. The major achievement of this 1960s work was the development of the Capital Asset Pricing Model (CAPM). In finance unlike in human genetics, a child can have four fathers. Fathers of CAPM all working independently were Treynor (1961, 1962), Sharpe (1964), Lintner (1965), and Mossin (1966).

Sharpe (1970), in particular, and Fama and Miller (1972), more cautiously, consider the Normal distributions essential to implementation of mean-variance techniques despite acknowledging that Stable distributions might better describe the real world.

CAPM is a model for pricing individual securities or entire portfolios based on a premium, the non-systematic risk component, over the market for the particular security risk class. All variants of the model have similar market presumptions and computational constraints. For purposes of this discussion, the constraints that concern us most are (1) the presumption that asset returns are Normally distributed random variables and (2) the appropriate measure of risk is the variance of asset returns.

During the 1960s in some academic circles, notably at the University of Chicago's Graduate School of Business, the choice of the best model for describing returns on risky assets was an open question, subject to debate, and wanting of empirical evidence. What empirical evidence there was often exhibited departures from Gaussian Normal presumptions, most notably skewness and kurtosis.

If it could be shown that departures from Normality were prevalent and statistically significant, then the mathematics of MPT would be inappropriate and inapplicable. As difficult as it might be to develop businesses in a non-Normal world, it would be even more difficult to teach the subject to the waves of students entering graduate schools of business in that era.

While Mandelbrot was satisfied with pencil and paper, others harnessed the power of computers for improved speed and accuracy. Fama and Roll (1968) produced computer-generated tables that facilitated estimation of Stable distribution parameters whose procedures for computation were explained in Fama and Roll (1971). To simplify the process, to reduce the computer time and expense needed to produce results, and to work within the limitations of the mathematics extant that did not permit exploration of non-symmetrical estimation across the entire parameter space, Fama and Roll presumed that distributions were symmetric, i.e., the Skewness Parameter was 0.0. To adherents of the Efficient Market Hypothesis, this symmetry presumption must have seemed reasonable and consistent with tenets of EMH.

While some research into stock price movements was conducted with the Fama and Roll tables, more research applied other distribution models and even combinations or overlays of other distribution models. The debate was lively albeit confined to academic circles. But, finance and investment history changed abruptly and dramatically in the early 1970s. The severe stock and bond market declines in 1973 and 1974 sent professional investment managers scrambling for explanations and, perhaps more importantly, for new techniques to navigate the troubled waters. Fundamental analysis, charting, and other popular techniques were seen as failures so the hope was to find a better way. Enter MPT.

Because there was doubt that mean-variance MPT methodologies were inferior to alternative methodologies, stock and bond investment managers readily adopted MPT methodologies to explain, analyze, and promote their tactical and strategic portfolio management programs.

Thus, further consideration of the nature of asset return distributions essentially disappeared from the field finance and resurfaced in the field of statistics.

McCulloch (1986) pushed the analysis of Stable distributionsi a step further by introducing a approach using easily obtainable quantiles of the distribution. The computations involve a few two-way interpolations between points in tables. McCulloch added three important improvements over the Fama and Roll estimates. First, he relaxed the constraint of symmetry. Second, he provided estimates of distribution skewness across the entire parameter space. Third, he provided estimates of standard errors of each of the Stable distribution parameters, thereby allowing researchers to express confidence about the degree to which sample distributions might differ from from the Normal distribution.

Stable Distributions in Real Estate

The degree to which Modern Portfolio Theory (MPT) and Efficient Markets Hypothesis (EMH) came to dominate the investment world was as remarkable as it was contrary to traditional thinking about how theories may be derived from empirical evidence. After nearly two decades of influence in the stock market, MPT and EMH were introduced to real estate without a shred of empirical justification. In Young and Graff (1996) the authors assert: "MPT and EMH seem to have been introduced into real estate to justify the use of particular statistical techniques and portfolio strategies rather than as a consequence of empirical analysis of investment return and risk characteristics. In science, the situation is generally reversed: theories are developed to explain observations."

McCulloch's quantile-based methodology has proven robust for the analysis of real estate return distributions in the United States (Young and Graff (1995), Brown (2000), and Young (2008)), Australia (Graff, Harrington, and Young (1997)), the United Kingdom (Young, Lee, and Devaney (2006)), and, most recently, in Germany (Richter, Thomas, and Fuss (2011)). As persuasive as these efforts appear, there is still much to be probed and researched before the somewhat startling opening sentence of the last mentioned paper can prevail among researchers and practitioners in the field. Richter, Thomas and Fuss say "One of the primary tenets of property investment research is that return distributions of direct real estate show 'fat tails,' and therefore deviate from normality."

In discussing the state of Maximum Likelihood Estimators (MLE), Young and Graff (1995) write: "While they are theoretically more accurate than the McCulloch estimators, in most economic applications the marginal increase in accuracy over the McCulloch estimators is more than outweighed by the far greater inconvenience of implementation. The only applications in which these estimators are likely to be important are those involving relatively small data sets having sample characteristic exponents close to 2.0."ii

That was then; this is now. MLE has surpassed McCulloch's estimators in ease of use for both academics and practicioners alike because fast, inexpensive computing is now ubiquitous. MLE applications have been available in the Fortran programming language for more than a decade and recently these routines have been added to the kernel of symbolic computing software packages like Mathematica.

The idea behind Maximum Likelihood Estimation (MLE) of distribution parameters is to derive the various parameters to maximize the probability that the parameters best describe the sample distribution. While seemingly simple in concept, MLE is numerically intense, which makes the availability of powerful computers for data processing necessary.

For those inclined toward empirical analysis of sample data, the advent of powerful software coupled with the availability of fine-grained performance data offer an opportunity to (1) examine the real world characteristics of real estate as an investable asset class, (2) probe for similarities and differences among discrete physical and functional dimensions, (3) develop strategies and tactics to take advantage of persistent similarities and differences, (4) test alternative measures of real estate risk that could be used to mitigate potential losses or enhance portfolio performance, and (5) abandon the mathematics of MPT altogether as a technical construct of finance that is unworkable in real estate.

For our purposes, perhaps the most useful byproduct of MLE's good statistical properties is its ability to provide confidence intervals to quantify uncertainty. With respect to McColluch's estimates of confidence, modern MLE measures are a substantial improvement in precision. In the results that follow, for example, the magnitude of the standard errors for the characteristic exponent from MLE are a between one-quarter and one-third the size of those produced by McCulloch's quantile model.

The data and analysis of this paper extend the research presented by originally by Young and Graff (1995) and expanded a bit by Young (2008). Now, with the passage of time across a wider range of macroeconomic conditions, this work can be extended to 2010 and can include a sufficiently large sample of Apartment properties from 1989 onwards. Also, this analysis departs from the two cited studies and others done by similar means in Australia and the U.K. by employing newer MLE methodology rather than McCulloch's quantile methodology.

By application of MLE, this paper tests empirically whether property return distributions have finite variance and are Gaussian Normal. The short answer is that they still are not over a broader range of property types with even more precision and greater statistical confidence.

Stable Distributions

Stable distributions appeal to the Generalized Central Limit Theorem axiom that if a distribution has a limiting distribution, it must be a member of the Stable class. Normal distributions are Stable and are the only Stable distributions with finite variance. The log characteristic functions of Stable distributions have the following form for cases where $\alpha \neq 1$:

$$\psi(t) = i\delta t - |\gamma t|^{\alpha}\left[1 - i\beta \operatorname{sgn}(t)\tan(\pi\alpha/2)\right] \qquad \text{(Equation 1)}$$

The four parameters α, β, γ, and δ in Equation (1) completely characterize the distribution.

The Characteristic Exponent α lies in the half-open interval (0,2] and measures the rate at which the tails of the density function decline to zero. The larger the value of the Characteristic Exponent α, the faster the tails shrink toward zero. When α=2.0, the distribution is Normal.

While the means (first moments) of Stable distributions with Characteristic Exponents α>1.0 do exist, variances (second moment [illegible] -i.e., are infinite—for those distributions with Characteristic Exponents α [illegible]

The Skewness Parameter β lies in the closed inte[illegible]s a measure of the asymmetry of the distribution. The closer the Charac[illegible]t α is to the upper limit of the permissible range, the less significance th[illegible]n terms of shifting the shape of the distribution away from the correspon[illegible]distribution. At the limit α=2.0, the Normal distribution, the Skewness P[illegible]mes irrelevant and all Stable distributions are symmetric.

The Scale Parameter γ lies in the open interval (0, [illegible]), and is a measure of the spread of the distribution. If α=2.0, the Scale Parameter γ is directly proportional to the standard deviation: $\gamma=\sigma/\sqrt{2}$. However, the Scale Parameter γ is finite for all Stable distributions, despite the fact that the standard deviation is infinite for all α<2.0. Thus, the Scale Parameter γ can be regarded as a generalization of the standard deviation.

The Location Parameter δ is a rough measure of the midpoint of the distribution. A change in δ simply shifts the graph of the distribution left or right, hence the term "location."

There are a number of parameterizations of stable laws (Nolan (1997, 1998, and 2005)). Two are predominant in financial applications. Nolan's S0 is useful in theoretical work as it is continuous in all four parameters; Nolan's S1 is often used because the location parameter is the mean.

Data Description

Individual property returns of institutional-grade U.S. commercial real estate are available from the National Council of Real Estate Investment Fiduciaries (NCREIF). Furthermore, the NCREIF Property Index has various sub-indices compiled along major geographical region, property type, and combined region and property type dimensions.

Reported returns are based on income and asset value changes (i.e., capital gains, realized or unrealized) as determined by periodic valuation by investment managers, institutional owners, or third-party appraisers. While quarterly and annual returns are available, we use only the annual total returns for the calendar years 1980 to 2010 disaggregated by four property types: Office, Retail, Industrial, and Apartment (Apartment data are only available in sufficient quantity from 1989 onwards). By using annual returns we avoid problems associated with autocorrelation or stale valuations in the sense that valuations are not conducted quarterly on all properties. For inclusion in the NCREIF Property Index, properties must be valued at least once per year.

For analytical purposes, each individual annual sample return rt in the NCREIF database has been replaced with its continuously compounded equivalent, ln(1+rt)

Further, only properties having four quarters of data in a given calendar year have been included.

Real Estate Return Model

A comparison of the data in the NCREIF Property Type sub-indices reveals significant differences among the annual returns. Our real estate market model assumes that expected variations in annual property returns due to differences in property type account for all of the differences in returns on properties in the NCREIF database.[iii]

We assume that the observed annual total return on each commercial property p during the calendar year *t* is of the following form:

$$R_t(p) = \mu_t(h(p)) + \varepsilon_t(p) \qquad \text{(Equation 2)}$$

where $h()$ is the property type (office, retail, or industrial), $\mu_t()$ is the expected total return during year *t* as a function of property type, and $\varepsilon_t(p)$ is a Stable (possibly, infinite-variance) random variable. In addition, we assume that, for each $t \geq 1980$, the $\varepsilon_t()$ are independent identically distributed random variables with Characteristic Exponent $\alpha_t > 1.0$ and zero mean, and that $\varepsilon_{t_1}(p_i)$ and $\varepsilon_{t_2}(p_j)$ are independent for all $t_1 \neq t_2$ and all *i* and *j*.[iv]

Under these assumptions, the random variable $\varepsilon_t(p)$ corresponds to the asset-specific investment risk of property *p* during period *t*, while the systematic and market sector real estate risk is described by the function $\mu_t(h(\ ))$.

Analytical Tools

There are a variety of commercial computer software products including Mathematica, MATLAB, Maple, SAS, and SPSS; an open-source, free application known as R; as well as spreedsheet applications like Excel and Numbers that contain statistical routines. Mathematica from Wolfram Research is a symbolic-logic computing software application that offers an extensive suite of statistical tools including Stable distributions. Independent applications written in different source code such as Fortran can be integrated with standard Mathematica tools via a sub-routine know as Mathlink. John Nolan's STABLE for Mathematica, DOS-based Fortran application that uses Mathlink is available at his stable distribution web site http://academic2.american.edu/~jpnolan/stable/stable.html.

The standard error results of this paper were computed with Nolan's STABLE application.

Tests and Results

Before fitting Stable distributions to the sample data, we corrected for possible extraneous data dispersion by reducing each annual return by the corresponding sample mean for that calendar year and property type (cf. Equation (2)). The means are shown in Exhibit 5 for purposes of completeness, but will not be needed in the subsequent discussion.

Using Mathematica's Maximum Likelihood Estimation routine we fit a Stable distribution to each set of residuals arranged by calendar year and property type. To test whether the parameters varied during the sample period, Stable parameters were estimated for sets composed of the residuals aggregated across calendar years and property types. These results are tabulated in Exhibit 1 and are displayed graphically together with 95% and 99% confidence intervals (for all years where standard errors could be ascertained) in Exhibit 2 for the parameter α. Confidence intervals are not shown for the graphed results for the parameters β and γ in Exhibits 3 and 4 (δ, the Location Parameter, is irrelevant as an estimator of the mean for our purposes because our analysis adjusts for the effect of time-varying means).

In the case of Characteristic Exponents αt estimated by calendar year and property type, 100% of the samples (for which standard errors could be computed) by property type were distinct statistically from 2.0, the Characteristic Exponent of the Normal distribution, with 99% confidence. In the case of residuals aggregated across property type (the first panel of Exhibit 1), all sample Characteristic Exponents αt were distinct from 2.0 with 99% confidence.

Exhibit 2 displays the sample Characteristic Exponents αt of both the aggregated and individual property type residuals. It appears that αt could be time-invariant. However, Exhibit 5 that shows graphical representations of these data, suggests that αt likely varies across property type. From Exhibit 1, for the entire 1980 to 2010 sample period, estimates of Characteristic Exponents together with their standard errors are 1.598 ±0.003 for all four property types combined, 1.652 ±0.005 for Office properties, 1.515 ±0.007 for Retail properties, 1.594 ±0.005 for Industrial properties, and over the 1989 to 2010 sample period, 1.664 ±0.007 for Apartment properties.

Exhibit 5 shows the Characteristic Exponent for each property type and the aggregate over the full 1980 to 2010 time period (note, however, that Apartment properties were not available prior to 1989) along with the 95% and 99% confidence bands. In the case of the Characteristic Exponent, Office and Apartment are statistically indistinguishable from one another while Retail and Industrial stand apart. These differences among property types deviate from conclusions of prior studies where more statistical similarities were observed. Perhaps the differences are simply the result of more samples, but a more likely explanation may be found in the improved precision and smaller standard errors attendant to analysis by MLE versus McCulloch quantile-based methodology.

Because this paper employs a MLE methodology that differs from prior work involving McCulloch's quantile methology, a comparison of the two is in order. Exhibit 6 presents a comparison for the Characteristic Exponent α and the Scale Parameter γ for the aggregate of all property types combined and for the office property type over the longest period available from prior work 1980 to 2003. While the year-by-year comparisons of the Characteristic Exponents estimated by each methodology differ, sometimes by seemingly large amounts, the means of the yearly Characteristic Exponents of by the two methodologies are statistically indistinguishable. Likewise, the comparisons of the Scale Parameters estimated by each methodology are statistically identical in their means and differ in small insignificant ways yearly. Comparisons form the industrial and office property types yield similar results although those are not shown for brevity.

The above analysis implies that over the sample period 1980 to 2010 (1) real estate investment risk was heteroscedastic for properties of a type and in the aggregate; (2) during virtually all sample subperiods and across property type, Stable infinite-variance skewed asset-specific risk functions with a Characteristic Exponent α of approximately 1.598 with a standard error of 0.003 modeled the observed distributions of return residuals better than Normally distributed risk candidates; and (3) property type differences in the Characteristic Exponent across property types are likely, which begs the question for further research into other dimensions along which distinct differences may emerge.

Conclusions: Part 1

The analysis in this study using Maximum Likelihood Estimation supports the conclusion that individual (continuously compounded) annual property returns in the NCREIF database are not Normally distributed for calendar years during the period 1980 to 2010,.

It also supports the conclusion that, for each calendar year *t* in that interval, there is a Stable infinite-variance distribution with Characteristic Exponent α_t such that the return on each property for year *t* can be represented as the average (mean) return for that year on properties of the same commercial type plus a random sample from the Stable distribution for that year, and furthermore that these samples are independent for distinct properties or calendar years. These Stable distributions can be considered to represent real estate asset-specific risk.

The data analysis strongly implies that both the skewness and magnitude of real estate asset-specific risk change over time, i.e., real estate risk is heteroscedastic with respect to both the amount of risk and the shape of the risk distribution.

As mentioned previously, the Scale Parameter γ measures the spread of the distribution analogous to the standard deviation of the Normal Distribution. Other measures are possible without resorting to extensive computation. For example, the semi-interquartile range closely approximates the Scale Parameter for Stable Distributions. So, just as standard deviation has been used aa a measure of stock market and real estate market return volatility, the Scale Parameter can serve the same purpose for a more generalized and empirically supportable distribution of returns. Because the MLE fit of the Scale Parameter as a measure of volatility does not involve squaring of the extreme returns as standard deviation does, the Scale Parameter is probably a more accurate measure than standard deviation.

Perhaps unsurprisingly, the Scale Parameter tends to spike in periods when transactions are relatively few in number such as the 1991-1992 and the 2008-20109 periods. When markets are weak, appraisers have few touchstones for estimating market discount rates or capitalization rates, the key drivers of discounted cash flow models that predominate in commercial income-producing property valuation. This lack of consensus about discount rates and capitalization rates increases the likelihood that valuations will span a wider range of estimates than valuations conducted in more active market environments. Thus, returns computed as changes in value from the previous period would naturally exhibit wider variations or more reported volatility.

Unlike suggestions in earlier studies using McCulloch's quantile approach, the MLE analysis of this study supports the conclusion that there is no single value for the Characteristic Exponent of asset-specific risk across property type. Nonetheless, all four NCREIF property types exhibit statistically significant departures from the Normal distribution at the 99% confidence level for all years of this study. Improvements in precision of MLE in Stable distribution parameter estimation over the quantile approach coupled with the greater precision of standard error estimates eliminate doubts that sample distributions might occasionally approximate the Normal distribution as shown in earlier studies.

Since most institutional real estate investors hold portfolios comprised of properties of two or more property types, estimates of Stable distribution parameters for the aggregage NCREIF data may be useful. For the aggregate of NCREIF Office, Retail, Industrial, and Apartment property returns, however, a statistical estimate for the Characteristic Exponent α together with a 99% confidence interval around this value is 1.598 ±0.007, based on a sample distribution of 70,631 annual property returns over the thirty-one-year sample period. This interval is so far removed from 2.0—the value for a Normal distribution—that it has profound implications for real estate portfolio management with respect to portfolio risk reduction and assemblage as described more thoroughly in earlier referenced publications by the authors.

The conclusions of this study reinforce the earlier conclusion that in institutional-grade real estate portfolios, the appropriate degree of risk reduction across multiple risk factors (locational, economic, etc.) could only be achieved by purchasing most of the institutional-grade properties in the U.S.—an obvious practical impossibility. This implies that institutional real estate portfolio management must be concerned with the asset-specific risk component of each property included in the portfolio with perhaps lesser consideration given to market/systematic and market-sector risk components. In street parlance, this means that your real estate eggs are always in one basket, so it behoves you to watch those individual eggs very carefully.

Conclusions: Part 2

Slowly, knowledge of the non-Normal characteristics of real estate return distributions has spawned inquiry into processes and applications beyond the hackneyed mean-variance models. Brown (2004) notes differences in skewness of returns between institutional-grade commercial property and direct, private investment property where investors can influence the outcome. Commercial property returns, subject to bond-like leases for a share of their total value estimates, often exhibit bond-like negative skewness. By contrast, Brown finds that direct property returns are more likely to be positively skewed leading to the conjecture that the value-added activities of owners is, on balance, a net positive for performance.

Coleman and Mansour (2005) propose a real estate asset allocation model using a noncentral Student-t distribution to overcome the constraints imposed by the mean-variance model. This model is a generalization of the popular Student-t model in that it allows for skewness and kurtosis typically found in real-world real estate return distributions.

Richter, Thomas and Fuss (2011) use McCulloch's quantile methodology to investigate real estate return distributions in Germany and add the dimension of within-country regional location to the mix. This same dimension was addressed using non-parametric tests in the U.K. by Devaney, Lee, and Young (2007).

Research into the heavy tail on the negative side of the return distributions has implications for risk management or mitigation, for capital adequacy requirements, and for liquidity, especially for institutions subject to the strictures of the Basel Accords and other portfolio owners or managers.

Brown and Young (2011) offer a middle-ground solution to measuring downside risk in real estate by means of so-called Coherent Risk Measures, an improvement both quantitatively and qualitatively over the more familiar Value At Risk (VaR) measures used in banking institutions.

Mathematica and tools built upon the platform collectively called Wolfram Demonstrations Project offer quick, easy, and graphically interesting ways to probe aspects of return distributions and examine their consequences. For example, Brown's "Forming the Efficient Frontier When Returns are Non-Normal" demonstration shows efficient frontiers generated Normal versus Stable distribution under user-selectable variations of Stable distribution parameters.[v]

Applications of Stable distribution parameters to real estate return distributions to date have generally involved proprietary or difficult-to-acquire databases. NCREIF data are available to members pursuant to a "disaggregation request." Investment Property Databank (IPD) data required processing by an employee of IPD but the data in raw form were unavailable to the other researchers on the study. These restrictive procedures encumber would-be researchers and doubtless limit the investigations that might afford more and varied insights into the nature and implications of real estate return distributions. We can hope that the efforts of those who have been able to penetrate the labyrinth thus far will convince database gatekeepers that openess and transparency can benefit the entire industry.

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Exhibit 1
MLE Stable Distribution Parameters for NCREIF Property Database
Log Annual Total Return Residuals & Mean Returns & Number of Properties

All Properties Combined:

Year or Period	α	β	γ	Mean Return	Number of Properties
2010	1.644 **	-0.366	0.074	0.111	5,455
2009	1.708 **	-0.729	0.094	-0.181	5,588
2008	1.691 **	-0.603	0.079	-0.064	5,552
2007	1.555 **	0.374	0.047	0.118	4,702
2006	1.556 **	0.375	0.054	0.137	4,019
2005	1.643 **	0.357	0.064	0.157	3,383
2004	1.668 **	-0.227	0.058	0.110	3,266
2003	1.574 **	-0.514	0.052	0.072	3,236
2002	1.544 **	-0.559	0.052	0.057	3,070
2001	1.416 **	-0.365	0.041	0.070	2,636
2000	1.398 **	0.261	0.040	0.113	2,292
1999	1.416 **	0.172	0.037	0.107	2,059
1998	1.531 **	0.517	0.047	0.139	1,885
1997	1.496 **	0.390	0.050	0.133	1,859
1996	1.474 **	0.076	0.048	0.104	1,925
1995	1.484 **	-0.259	0.052	0.090	1,727
1994	1.433 **	-0.321	0.056	0.073	1,707
1993	1.446 **	-0.667	0.068	0.016	1,848
1992	1.500 **	-0.897	0.078	-0.041	1,907
1991	1.593 #	-0.996	0.084	-0.061	1,825
1990	1.464 **	-0.836	0.062	-0.004	1,595
1989	1.451 **	-0.556	0.062	0.038	1,366
1988	1.525 **	-0.570	0.066	0.055	1,221
1987	1.482 **	-0.703	0.076	0.040	1,146
1986	1.647 **	-0.828	0.063	0.061	1,071
1985	1.696 **	-0.406	0.057	0.098	937
1984	1.472 **	0.095	0.050	0.118	894
1983	1.466 **	-0.171	0.058	0.102	877
1982	1.476 **	-0.095	0.054	0.086	685
1981	1.356 **	0.645	0.060	0.160	507
1980	1.275 **	0.817	0.046	0.155	391
1980-10	1.598 **	-0.235	0.064	0.046	70,631
std. dev.	0.003				

Exhibit 1 (continued)
MLE Stable Distribution Parameters for NCREIF Property Database
Log Annual Total Return Residuals & Mean Returns & Number of Properties

Office Properties:

Year or Period	α	β	γ	Mean Return	Number of Properties
2010	1.587 **	-0.284	0.079	0.068	1,288
2009	1.734 **	-0.793	0.111	-0.216	1,386
2008	1.689 **	-0.729	0.087	-0.077	1,388
2007	1.574 **	0.427	0.063	0.139	1,087
2006	1.542 **	0.331	0.064	0.140	970
2005	1.463 **	0.202	0.066	0.149	909
2004	1.610 **	-0.456	0.064	0.083	923
2003	1.463 **	-0.744	0.058	0.033	962
2002	1.413 **	-0.805	0.060	0.012	932
2001	1.477 **	-0.553	0.052	0.048	830
2000	1.285 **	0.344	0.042	0.119	661
1999	1.534 **	0.362	0.043	0.112	591
1998	1.434 **	0.818	0.056	0.171	498
1997	1.590 **	0.604	0.069	0.181	406
1996	1.715 **	0.310	0.065	0.128	431
1995	1.437 **	-0.325	0.069	0.072	383
1994	1.475 **	-0.447	0.080	0.051	414
1993	1.423 **	-0.809	0.083	-0.032	468
1992	1.478 #	-0.968	0.099	-0.115	448
1991	1.440 #	-1.000	0.107	-0.153	454
1990	1.388 #	-0.957	0.078	-0.076	412
1989	1.246 **	-0.768	0.069	-0.022	385
1988	1.449 **	-0.816	0.071	-0.001	370
1987	1.246 #	-0.901	0.078	-0.023	355
1986	1.474 #	-0.928	0.067	0.020	339
1985	1.765 **	-0.759	0.065	0.068	287
1984	1.578 **	-0.115	0.059	0.102	249
1983	1.399 **	-0.324	0.060	0.100	237
1982	1.563 **	0.340	0.057	0.101	172
1981	1.468 **	0.820	0.061	0.173	92
1980	1.478 #	1.000	0.050	0.153	65
1980-10	1.652 **	-0.425	0.078	0.028	18,392
std. dev.	0.005				

Exhibit 1 (continued)
MLE Stable Distribution Parameters for NCREIF Property Database
Log Annual Total Return Residuals & Mean Returns & Number of Properties

Retail Properties:

Year or Period	α	β	γ	Mean Return	Number of Properties
2010	1.522 **	-0.474	0.066	0.083	908
2009	1.627 **	-0.658	0.079	-0.140	831
2008	1.660 **	-0.657	0.076	-0.062	888
2007	1.432 **	0.499	0.036	0.107	721
2006	1.543 **	0.653	0.037	0.118	572
2005	1.658 **	0.580	0.056	0.161	457
2004	1.649 **	0.335	0.052	0.172	462
2003	1.565 **	-0.122	0.042	0.129	427
2002	1.364 **	0.075	0.037	0.106	454
2001	1.204 **	-0.404	0.030	0.072	450
2000	1.227 **	-0.180	0.032	0.090	443
1999	1.445 **	0.000	0.035	0.104	408
1998	1.412 **	-0.000	0.038	0.115	405
1997	1.207 **	-0.130	0.038	0.094	443
1996	1.064 **	-0.300	0.035	0.060	511
1995	1.172 **	-0.625	0.043	0.040	383
1994	1.172 **	-0.320	0.038	0.058	373
1993	1.198 **	-0.497	0.046	0.034	417
1992	1.341 #	-0.933	0.055	-0.006	385
1991	1.522 **	-0.812	0.064	-0.023	378
1990	1.251 **	-0.564	0.035	0.048	279
1989	1.314 **	-0.156	0.045	0.073	221
1988	1.666 **	-0.104	0.060	0.106	202
1987	1.634 **	-0.296	0.054	0.105	194
1986	1.534 **	0.135	0.043	0.112	179
1985	1.476 **	0.087	0.046	0.116	168
1984	1.565 **	0.738	0.041	0.131	165
1983	1.467 **	-0.000	0.049	0.116	157
1982	1.355 **	-0.239	0.049	0.075	116
1981	1.545 **	-0.318	0.053	0.090	98
1980	1.230 **	0.569	0.036	0.128	74
1980-09	1.515 **	-0.238	0.053	0.056	12,169
std. dev.	0.007				

Exhibit 1 (continued)
MLE Stable Distribution Parameters for NCREIF Property Database
Log Annual Total Return Residuals & Mean Returns & Number of Properties

Industrial Properties:

Year or Period	α	β	γ	Mean Return	Number of Properties
2010	1.637 **	-0.445	0.074	0.074	1,994
2009	1.661 **	-0.797	0.095	-0.190	2,054
2008	1.589 **	-0.513	0.071	-0.058	1,946
2007	1.658 **	0.252	0.046	0.119	1,826
2006	1.637 **	0.339	0.057	0.144	1,684
2005	1.767 **	-0.072	0.069	0.161	1,344
2004	1.649 **	-0.513	0.060	0.107	1,204
2003	1.514 **	-0.495	0.053	0.078	1,162
2002	1.360 **	-0.613	0.044	0.064	996
2001	1.287 **	-0.225	0.034	0.085	762
2000	1.252 **	0.392	0.035	0.125	650
1999	1.214 **	0.068	0.033	0.107	594
1998	1.441 **	0.633	-0.043	0.145	575
1997	1.479 **	0.658	0.050	0.146	617
1996	1.360 **	0.382	0.039	0.120	636
1995	1.538 **	-0.065	0.045	0.117	656
1994	1.536 **	-0.341	0.053	0.078	630
1993	1.303 **	-0.781	0.061	0.006	695
1992	1.293 #	-0.903	0.071	-0.036	850
1991	1.413 #	-0.915	0.070	-0.037	795
1990	1.202 **	-0.769	0.051	0.001	740
1989	1.326 **	-0.419	0.053	0.058	663
1988	1.398 **	-0.388	0.058	0.071	649
1987	1.439 **	-0.521	0.070	0.056	597
1986	1.641 **	-0.791	0.060	0.070	553
1985	1.676 **	-0.355	0.054	0.110	482
1984	1.403 **	0.072	0.049	0.122	480
1983	1.472 **	-0.139	0.059	0.099	483
1982	1.395 **	-0.139	0.051	0.082	397
1981	1.096 **	0.622	0.051	0.178	317
1980	1.163 **	0.867	0.044	0.163	252
1980-09	1.594 **	-0.298	0.063	0.052	27,283
std. dev.	0.005				

Exhibit 1 (continued)
MLE Stable Distribution Parameters for NCREIF Property Database
Log Annual Total Return Residuals & Mean Returns & Number of Properties

Apartment Properties:

Year or Period	α	β	γ	Mean Return	Number of Properties
2010	1.787 **	-0.100	0.072	0.159	1,265
2009	1.798 **	-0.573	0.082	-0.158	1,317
2008	1.843 **	-0.727	0.081	-0.062	1,330
2007	1.477 **	0.472	0.042	0.101	1,068
2006	1.553 **	0.430	0.052	0.131	793
2005	1.458 **	0.853	0.055	0.155	673
2004	1.653 **	0.692	0.044	0.111	677
2003	1.650 **	0.054	0.042	0.081	685
2002	1.755 **	0.149	0.047	0.077	688
2001	1.748 **	-0.347	0.044	0.082	594
2000	1.670 **	0.246	0.040	0.108	538
1999	1.483 **	0.349	0.033	0.103	466
1998	1.642 **	0.608	0.041	0.117	407
1997	1.487 **	0.437	0.036	0.109	393
1996	1.555 **	0.444	0.039	0.108	347
1995	1.474 **	0.627	0.037	0.118	305
1994	1.673 **	0.136	0.053	0.114	290
1993	1.260 **	0.221	0.046	0.099	268
1992	1.177 **	-0.738	0.039	0.027	224
1991	1.372 **	-0.847	0.059	-0.017	198
1990	1.793 #	-1.000	0.052	0.062	164
1989	1.497 **	-0.225	0.049	0.061	97
1989-09	1.664 **	0.114	0.056	0.047	12,787
std. dev.	0.007				

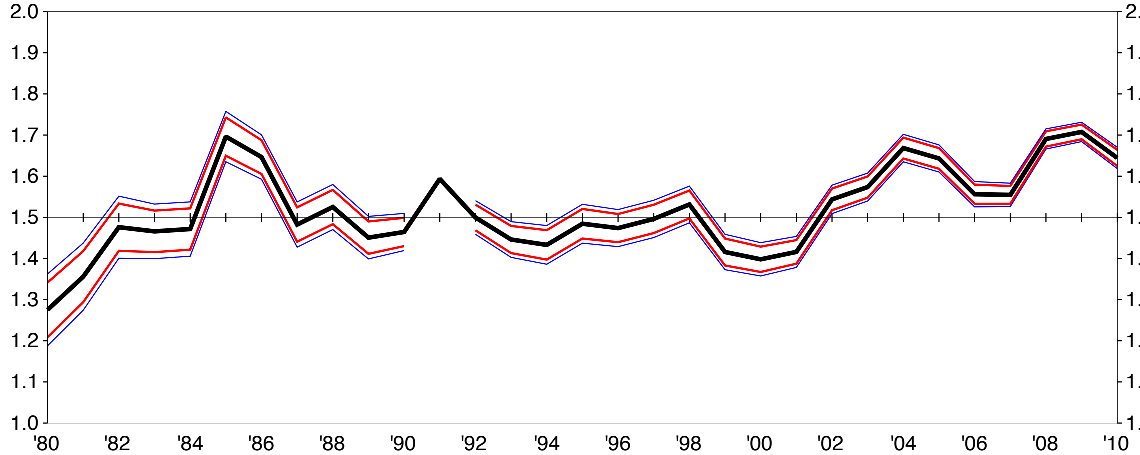
Statistically significant confidence of non-Normality α ≠ 2.0):
** = 99% confidence
# = indeterminate

α is the Characteristic Exponent, and only equals 2.0 for the Normal distribution
β is the Skewness Parameter in the range -1.0 to +1.0
γ is the (positive) Scale Parameter which measures the spread of the distribution about δ

Note: The means are shown in Exhibit 1 simply for purposes of completeness.

Exhibit 2
Characteristic Exponent "Alpha" of Distributions of Log Annual Total Return Residuals
NCREIF 1980 to 2010

All Properties (exclusive of Apartment Properties 1980 to 1988)
(bands indicate plus and minus 95% and 99% confidence intervals)



Office Properties
(bands indicate plus and minus 95% and 99% confidence intervals)

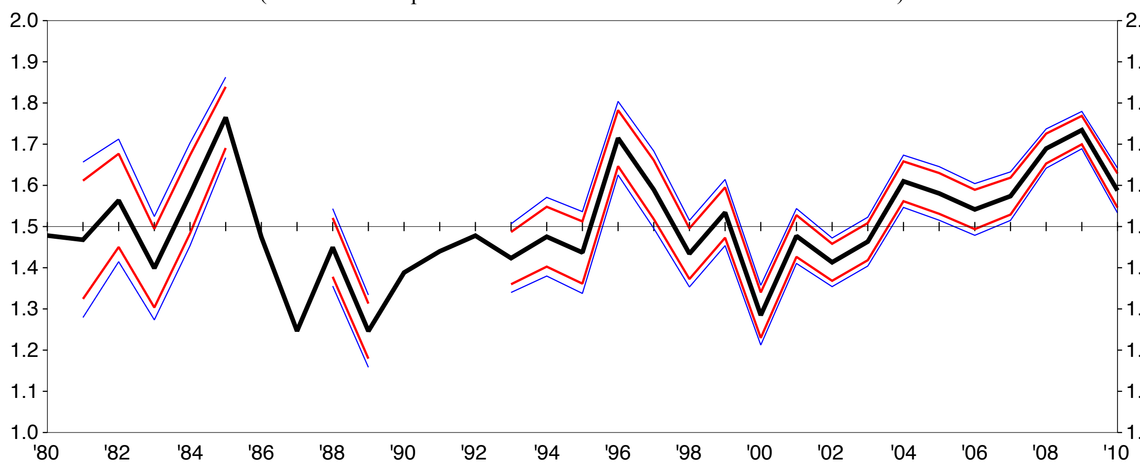
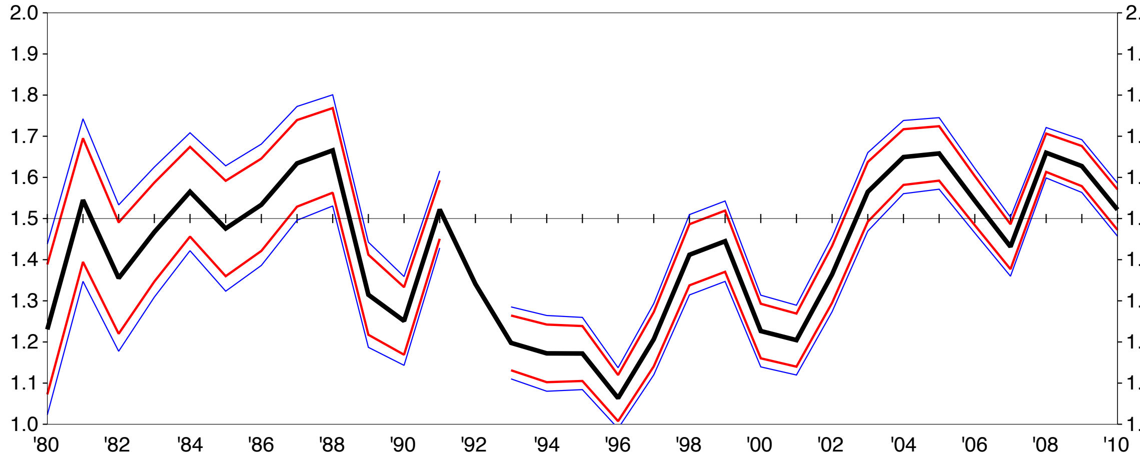


Exhibit 2 (continued)
Characteristic Exponent "Alpha" of Distributions of Log Annual Total Return Residuals
NCREIF 1980 to 2010

Retail Properties
(bands indicate plus and minus 95% and 99% confidence intervals)



Industrial Properties
(bands indicate plus and minus 95% and 99% confidence intervals)

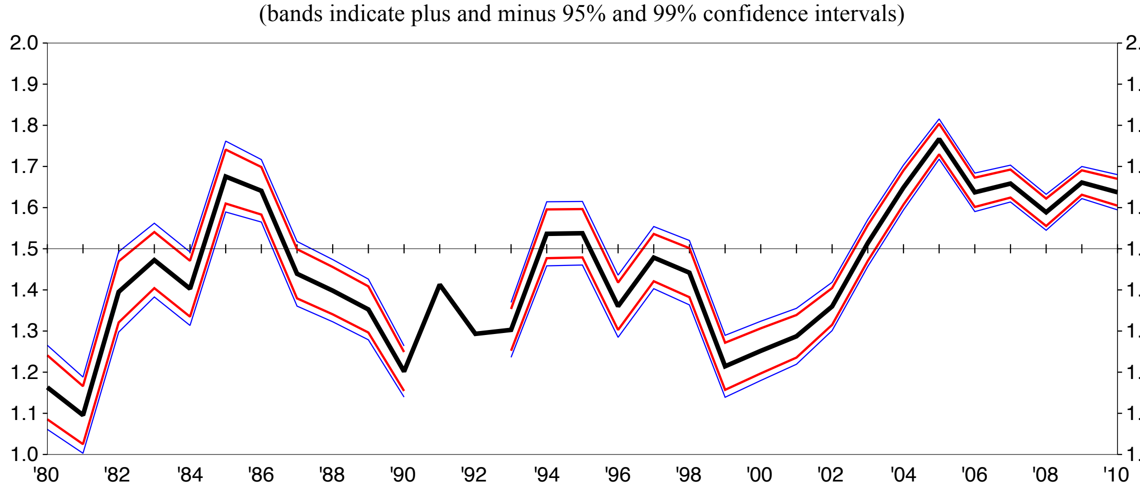


Exhibit 2 (continued)
Characteristic Exponent "Alpha" of Distributions of Log Annual Total Return Residuals
NCREIF 1980 to 2010

Apartment Properties
(bands indicate plus and minus 95% and 99% confidence intervals)

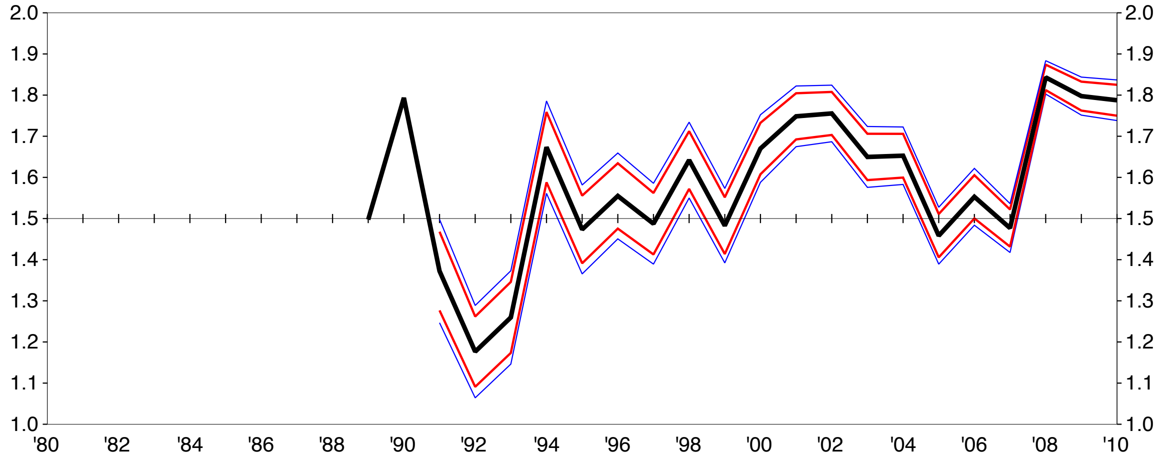


Exhibit 3
Skewness Parameter "Beta" of Distributions of Log Annual Total Return Residuals
NCREIF 1980 to 2010

All Properties (exclusive of Apartment Properties 1980 to 1988)

Exhibit 4
Scale Parameter "Gamma" of Distributions of Log Annual Total Return Residuals
NCREIF 1980 to 2010

All Properties (exclusive of Apartment Properties 1980 to 1988)

Exhibit 5
Characteristic Exponent "Alpha" of Distributions of Log Annual Total Return Residuals
By Property and in the Aggregate
NCREIF 1980 to 2010

All Properties (exclusive of Apartment Properties 1980 to 1988)
(bands indicate plus and minus 95% and 99% confidence intervals)

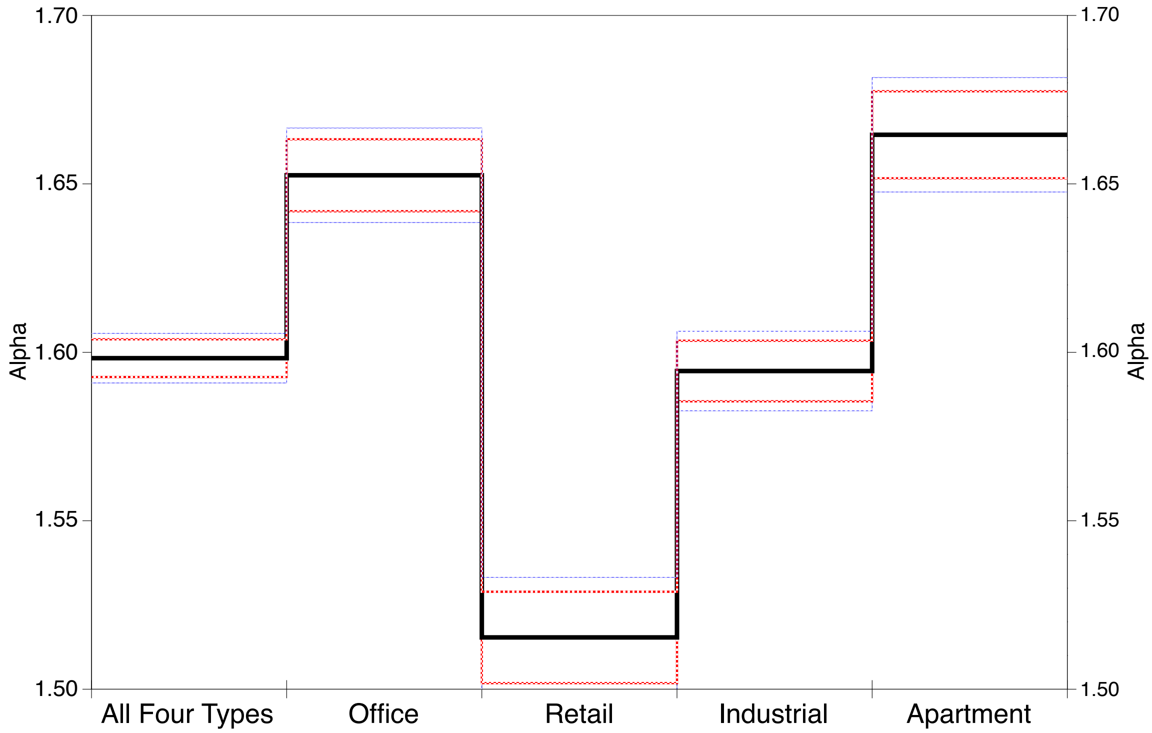


Exhibit 6
Comparison of Stable Distribution Parameters "Alpha" and "Gamma"
Computed by McCulloch's Quantile and MLE Methods

All Properties Combined:

		Alpha			Gamma	
Year	McCulloch	MLE	diff	McCulloch	MLE	diff
2003	1.428	1.574	0.146	0.054	0.052	–0.002
2002	1.422	1.544	0.122	0.050	0.052	0.002
2001	1.297	1.416	0.119	0.041	0.041	0.000
2000	1.269	1.398	0.129	0.039	0.040	0.001
1999	1.325	1.416	0.091	0.037	0.037	0.000
1998	1.466	1.531	0.065	0.051	0.047	–0.004
1997	1.432	1.496	0.064	0.055	0.050	–0.005
1996	1.361	1.474	0.113	0.048	0.048	0.000
1995	1.372	1.484	0.112	0.050	0.052	0.002
1994	1.263	1.433	0.170	0.052	0.056	0.004
1993	1.485	1.446	–0.039	0.069	0.068	–0.001
1992	1.594	1.500	–0/094	0.088	0.078	–0.010
1991	1.616	1.593	–0.023	0.088	0.084	–0.004
1990	1.351	1.464	0.113	0.060	0.062	0.002
1989	1.332	1.451	0.119	0.060	0.062	0.002
1988	1.449	1.525	0.076	0.063	0.066	0.003
1987	1.419	1.482	0.063	0.072	0.076	0.004
1986	1.452	1.647	0.195	0.057	0.063	0.006
1985	1.452	1.696	0.244	0.051	0.057	0.006
1984	1.325	1.472	0.147	0.047	0.050	0.003
1983	1.327	1.466	0.139	0.053	0.058	0.005
1982	1.358	1.476	0.118	0.051	0.054	0.003
1981	1.318	1.356	0.038	0.060	0.060	0.000
1980	1.486	1.275	–0.211	0.054	0.046	–0.008
mean	1.400	1.484	0.084	0.056	0.057	0.000
std dev	0.090	0.087	0.094	0.013	0.012	0.004

Exhibit 6 (continued)
Comparison of Stable Distribution Parameters "Alpha" and "Gamma"
Computed by McCulloch's Quantile and MLE Methods

Office Properties:

		Alpha			Gamma	
Year	McCulloch	MLE	diff	McCulloch	MLE	diff
2003	1.412	1.463	0.051	0.059	0.058	–0.001
2002	1.509	1.413	0.051	0.063	0.060	–0.003
2001	1.443	1.477	0.034	0.052	0.052	0.000
2000	1.265	1.285	0.020	0.043	0.042	–0.001
1999	1.495	1.534	0.039	0.043	0.043	0.000
1998	1.717	1.434	–0.283	0.066	0.056	–0.010
1997	1.754	1.590	–0.164	0.075	0.069	–0.006
1996	1.595	1.715	0.129	0.063	0.065	0.002
1995	1.367	1.437	0.070	0.066	0.069	0.003
1994	1.422	1.475	0.053	0.081	0.080	–0.001
1993	1.468	1.423	–0.045	0.087	0.083	–0.004
1992	1.551	1.478	–0.073	0.103	0.099	–0.004
1991	2.000	1.440	–0.560	0.138	0.107	–0.031
1990	1.459	1.388	–0.071	0.084	0.078	–0.006
1989	1.479	1.246	–0.233	0.081	0.069	–0.012
1988	1.544	1.449	–0.095	0.082	0.071	–0.011
1987	1.304	1.246	–0.058	0.081	0.078	–0.003
1986	1.413	1.474	0.061	0.066	0.067	0.001
1985	1.471	1.765	0.294	0.059	0.065	0.006
1984	1.334	1.578	0.244	0.050	0.060	0.010
1983	1.288	1.399	0.111	0.053	0.060	0.007
1982	1.683	1.563	–0.120	0.059	0.059	0.002
1981	1.520	1.468	–0.052	0.063	0.061	–0.002
1980	2.000	1.478	–0.522	0.063	0.050	–0.013
mean	1.521	1.468	–0.053	0.070	0.067	–0.003
std dev	0.189	0.119	0.196	0.020	0.015	0.008

[i] For a comprehensive annotated bibliography of published articles on Stable distributions and its application to various areas of finance, economics, and engineering see Nolan (2003, revised 2011).

[ii] There is also a problem when the Characteristic Exponent approaches 1.0.

[iii] Alternatively, we could have broken down returns by major geographic region. However, we believe that property type is the superior cut, because it is more likely that investment characteristics of commercial property differ for properties with different drivers of economic performance and different lease structures than for properties with the same economic functional attributes simply situated in different parts of the U.S. The free flow of institutional real estate investment capital across the country over the past forty years has homogenized transient differences in investment characteristics across geographical regions for property of the same type.

[iv] The assumption that $\alpha_t > 1.0$ guarantees that the mean of $\varepsilon_t(p)$ exists.

[v] The astute observer will immediately recognize a paradox in that efficient frontier graphics constitute a parametric plot that requires a covariance matrix. If Stable distributions have no variance, they can have no covariances. One must remember, however, that Stable distributions lack a variance in the limit. All finite samples have a variance that can be calculated. The demonstration illustrates the shape of the “frontier” using samples that are presumed to be drawn from a Stable population having parameters supplied by the user. The demonstration is located at:

http://demonstrations.wolfram.com/FormingTheEfficientFrontierWhenReturnsAreNonNormal/.